A soft-photon theorem for the Maxwell-Lorentz system

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joint work with Wojciech Dybalski[2]

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Fact

Local gauge symmetry in QED/ED implies

$$\phi(n) := \lim_{r\to\infty} r^2 n \cdot E(nr), \quad n \in S^2$$

(spacelike asymptotic flux)

is conserved.

Goal: Study for classical ED
asymptotic constants of motion $\leftrightarrow$ soft-photon theorem

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   - Asymptotic constants of motion
   - Soft-photon theorem
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Dynamics of a particle with charge e and radial "smeared" charge distribution $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ is determined by

Definition (Maxwell-Lorentz equations)

$$\begin{aligned} \nabla \cdot E(x,t) &= e\varphi\left(x - q(t)\right), \\ \nabla \cdot B(x,t) &= 0, \\ \partial_t E(x,t) &= \nabla \times B(x,t) - e\varphi\left(x - q(t)\right) v(t), \\ \partial_t B(x,t) &= -\nabla \times E(x,t), \\ \frac{d}{dt}\{m\gamma v(t)\} &= e\left\{E_\varphi\left(q(t),t\right) + v(t) \times B_\varphi\left(q(t),t\right)\right\}, \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-v^2(t)}}$ and $F_\varphi(x,t) := \left(F(\cdot,t) * \varphi\right)(x)$.

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- Rewrite as a generalized equation $\frac{d}{dt}Y(t) = F(Y(t)), Y(0) = Y^0$.
- Introduce phase space set $\mathcal{M}$ of the electric magnetic fields and trajectory.

Theorem (A. Komech, H. Spohn, 2000)

Let $Y^0 = (E^0, B^0, q^0, v^0) \in \mathcal{M}$. Then the integral equation associated with the equations of motion,

$$Y(t) = Y^0 + \int_0^t F(Y(s))\, ds,$$

has a unique solution $Y(t) = (E(\cdot,t), B(\cdot,t), q(t), v(t)) \in \mathcal{M}$ for all $t \in \mathbb{R}_{\geq 0}$, which is continuous in t and satisfies $Y(0) = Y^0$.

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Remark: Uniqueness and existence of solutions holds also for $t < 0$.

For $t \geq 0$:
$$E(t) = \partial_t G_{\text{ret},t} * E^0 + \nabla \times \left(G_{\text{ret},t} * B^0 \right) - \int_0^t \text{d}s \left\{ \nabla G_{\text{ret},t-s} * \rho(s) + \partial_t G_{\text{ret},t-s} * j(s) \right\}$$

where $G_{\text{ret},t}(x) := \frac{\theta(t)}{4\pi t} \delta(|x| - t)$.

For $t < 0$:
$$E(t) = \partial_t (-G_{\text{adv},t}) * E^0 + \nabla \times \left(-G_{\text{adv},t} * B^0 \right) - \int_0^t \text{d}s \left\{ \nabla G_{\text{adv},t-s} * \rho(s) + \partial_t G_{\text{adv},t-s} * j(s) \right\}$$

where $G_{\text{adv},t} := -\frac{\theta(-t)}{4\pi t} \delta(|x| + t) \equiv G_{\text{ret},-t}(x)$.

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For $t \in \mathbb{R}$:

$$E(t) = \partial_t G_t * E^0 + \nabla \times \left(G_t * B^0 \right) - \int_0^t \mathrm{d}s \left\{ \nabla G_{\text{ret/adv},t-s} * \rho(s) + \partial_t G_{\text{ret/adv},t-s} * j(s) \right\}$$

where $G_t := G_{\text{ret},t} - G_{\text{adv},t}$.

Example (Charge solitons/Travelling waves)

The solutions traveling with constant velocity $|v| < 1$ and starting at $q \in \mathbb{R}^3$ are uniquely given by

$$Y(t) = (E_v(\cdot - q - vt), B_v(\cdot - q - vt), q + vt, v) \in \mathcal{M},$$

where

$$B_v(x) := -v \times \nabla \phi_{v\varphi}(x),$$
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and

$$\phi_v(x) := \frac{e}{4\pi\sqrt{(x/\gamma)^2 + (v \cdot x)^2}}, \quad \phi_{v\varphi}(x) := (2\pi)^{-3/2}\phi_v * \varphi(x)$$

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For any $\sigma \in [0,1]$ define $\mathcal{M}^\sigma \subseteq \mathcal{M}$ by the condition

$$|E(x,t)| + |B(x,t)| + |x|\left(|\nabla E(x,t)| + |\nabla B(x,t)|\right) \leq \frac{C}{|x|^{1+\sigma}}$$

for all $|x| > R$ and $C, R > 0$.

Theorem (A. Komech, H. Spohn, V. Imaikin, 2000)

If $|e| \leq \overline{e}$ holds for a suitable $\overline{e}$ and $Y(0) \in \mathcal{M}^\sigma$, then the following statements are true:

1. $\exists C > 0 : \forall t \in \mathbb{R} : \qquad |\dot{v}(t)| \leq C(1+|t|)^{-1-\sigma}$
2. *There exist scattering fields* $Z_{sc}(t) := (E_{sc}(\cdot,t), B_{sc}(\cdot,t)) \subset \mathcal{M}$ *such that*

$$E(t) - E_{v(t)}(\cdot - q(t)) \xrightarrow[\|\cdot\|_{L^2}]{t\to\pm\infty} E_{sc,\pm}(t),$$

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Let F be the Faraday (field strength) tensor with corresp. Fourier transform $\hat{F}$.

In the situation of the previous theorem:

Theorem (W. Dybalski, D.V.H., 2019)

The limit $\mathfrak{F}(\hat{x}, t) := \lim_{|x|\to\infty} |x|^2 F(x, t)$ *exists for any* $t \in \mathbb{R}$ *if* $\mathfrak{F}(\hat{x}, 0)$ *exists. In particular, it holds that* $\mathfrak{F}$ *is conserved, i.e.*

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In this talk:

Theorem (W. Dybalski, D.V.H., 2019)

The limit $\mathcal{F}(\hat{k},t) := \lim_{|k|\to 0} |k|\hat{F}(k,t)$ *exists for any* $t \in \mathbb{R}$ *if* $\mathcal{F}(\hat{k},0)$ *exists. In particular, it holds that* $\mathcal{F}$ *is conserved, i.e.*

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Sketch of proof: Here, only for electric component $\mathcal{E}(\hat{k},t) = \lim_{|k|\to 0} |k|\hat{E}(k,t)$ and $t \geq 0$.

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Proof (sketch): From proof of the previous theorem:

$$\begin{aligned} E(x,t) = {} & \partial_t G_t * \left[E(\cdot,0) - E_{v(0)}(\cdot - q(0)) \right](x) \\ & + \nabla \times \left\{ G_t * \left[E(\cdot,0) - E_{v(0)}(\cdot - q(0)) \right](x) \right\} \\ & - \int_0^t ds \Big[\partial_\tau G_\tau \big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) E_v(\cdot - q(s))(x) \\ & \quad + \nabla \times \left\{ G_\tau \big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) B_v(\cdot - q(s))(x) \right\} \Big] \\ & + E_v(x - q(t)) \end{aligned}$$

in terms of the soliton fields and the retarded propagator $G_t(x) = \frac{1}{4\pi t}\delta(|x| - t)$.

Proof (sketch): Use the distributional Fourier transform

$$\begin{aligned}\hat{E}(k,t) = \cos(|k|t)\left[\hat{E}(k,0) - \hat{E}_{v(0)}(k)e^{ikq(0)}\right] \\ + i\hat{k} \times \left\{\sin(|k|t)\left[\hat{E}(k,0) - \hat{E}_{v(0)}(k)e^{ikq(0)}\right]\right\} \\ - \int_0^t ds\Big[\cos(|k|(t-s))\,(\dot{v}(s)\cdot\nabla_v)\hat{E}_v(k)e^{ikq(s)} \\ + i\hat{k} \times \left\{\sin(|k|(t-s))\,(\dot{v}(s)\cdot\nabla_v)\hat{B}_v(k)e^{ikq(s)}\right\}\Big] \\ + \hat{E}_v(k)e^{ikq(s)}.\end{aligned}$$

Proof (sketch): Thus, taking the limit yields for $v \equiv v(t)$

$$\begin{aligned}
\mathcal{E}(\hat{k}, t) &= \mathcal{E}(\hat{k}, 0) - \mathcal{E}_{v_0}(\hat{k}) + \mathcal{E}_v(\hat{k}) \\
&\quad - \lim_{|k| \to 0} \int_0^t ds \Big[\cos(|k|(t-s)) \, (\dot{v}(s) \cdot \nabla_v) |k| \hat{E}_v(k) e^{ikq(s)} \\
&\qquad + ik \times \Big\{ \sin(|k|(t-s)) \, (\dot{v}(s) \cdot \nabla_v) \hat{B}_v(k) e^{ikq(s)} \Big\} \Big] \\
&= \mathcal{E}(\hat{k}, 0) - \mathcal{E}_{v_0}(\hat{k}) - \int_0^t ds \left[(\dot{v}(s) \cdot \nabla_v) \mathcal{E}_v(\hat{k}) \right] + \mathcal{E}_v(\hat{k}) \\
&= \mathcal{E}(\hat{k}, 0).
\end{aligned}$$

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In the situation of the previous theorem:

Theorem (W. Dybalski, D.V.H., 2019)

The quantities

$$\mathcal{F}_{sc,\pm}(\hat{k}) = \lim_{|k|\to 0} |k| F_{sc,\pm}(k,t),$$

$$\mathcal{F}_{v_{\pm\infty}}(\hat{k}) = \lim_{|k|\to 0} |k| F_{v_{\pm\infty}}(k,t)$$

are well-defined and are related with each other via:

$$\mathcal{F}_{sc,+}(\hat{k}) + \mathcal{F}_{v_{+\infty}}(\hat{k}) = \mathcal{F}_{sc,-}(\hat{k}) + \mathcal{F}_{v_{-\infty}}(\hat{k}).$$

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Proof (sketch): (here, only for electric component)

$$E(x,t) - E_{sc,+}(x,t) = \int_t^\infty ds \left[\partial_\tau G_\tau \big|_{\tau = t-s} * (\dot{v}(s) \cdot \nabla_v) E_v(\cdot - q(s))(x) + \nabla \times \left\{ G_\tau \big|_{\tau=t-s} * (\dot{v}(s)\cdot \nabla_v) B_v(\cdot - q(s))(x) \right\} \right] + E_{v(t)}(x - q(t)).$$

Thus, we find similarly to the previous proof that

$$\mathcal{E}(\hat{k}) - \mathcal{E}_{v(t)}(\hat{k}) - \mathcal{E}_{sc,+}(\hat{k},t) = \mathcal{E}_{v_\infty}(\hat{k}) - \mathcal{E}_{v(t)}(\hat{k})$$

$$\implies \mathcal{E}(\hat{k}) = \mathcal{E}_{sc,+}(\hat{k}) + \mathcal{E}_{v_{+\infty}}(\hat{k}) = \mathcal{E}_{sc,-}(\hat{k}) + \mathcal{E}_{v_{-\infty}}(\hat{k}).$$

Proof (sketch): (here, only for electric component)

$$\begin{aligned} &E(x,t) - E_{\mathrm{sc},+}(x,t) \\ &= \int_t^\infty ds \Big[\partial_\tau G_\tau\big|_{\tau=t-s} * (\dot v(s)\cdot\nabla_v) E_v(\cdot - q(s))(x) \\ &\qquad + \nabla \times \Big\{ G_\tau\big|_{\tau=t-s} * (\dot v(s)\cdot\nabla_v) B_v(\cdot - q(s))(x) \Big\} \Big] \\ &\quad + E_{v(t)}(x - q(t)). \end{aligned}$$

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We showed that

$$\mathcal{E}_{sc,+}(\hat{k}) - \mathcal{E}_{sc,-}(\hat{k}) = -\left(\mathcal{E}_{v_{+\infty}}(\hat{k}) - \mathcal{E}_{v_{-\infty}}(\hat{k})\right).$$

and thus for any $k \in \mathbb{R}^3 \setminus \{0\}$ and longitudinal initial data

$$\hat{E}_{sc,+}(k) = -\left(P_{tr}\hat{E}_{v_\infty}\right)(\hat{k}) + R(k),$$

where $R \in o\left(1/|k|\right)$ and P_{tr} is the transverse projection w.r.t. $\hat{k} = k/|k|$.

$$\implies \mathcal{E}_{sc,+} = \lim_{|k|\to 0} |k|\hat{E}_{sc,+}(k,t)$$

$$= -\frac{ie}{(2\pi)^{3/2}}\left(\frac{\left(P_{tr}(\hat{k})v_\infty\right)\left(\hat{k}\cdot v_\infty\right)}{1-(\hat{k}\cdot v_\infty)^2}\right)$$

We showed that

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For which $w \in \mathbb{C}^3$

$$\lim_{k\to 0} |k| \langle w | \hat{\mathrm{E}}(k,t) | w \rangle = \lim_{|k|\to 0} |k| \hat{E}_{\mathrm{sc},+}(k,t) \qquad \text{for } t \geq 0$$

where

$$|w\rangle := \exp\left(\sum_{\lambda=\pm} \int \mathrm{d}^3 k \, \{ w(k) \cdot \varepsilon_\lambda(k) a^*_\lambda(k) - \mathrm{h.c.} \} \right) |0\rangle,$$

$$\hat{\mathrm{E}}(k,t) := \sum_{\lambda=\pm} \sqrt{\frac{|k|}{2}} \left(i\varepsilon_\lambda(k) e^{-i|k|t} a_\lambda(k) - i\varepsilon_\lambda(-k) e^{i|k|t} a^*_\lambda(-k) \right) \quad ?$$

$$\Longrightarrow \quad w(k) = -\frac{e\hat{\varphi}(k)}{\sqrt{2}|k|^{3/2}} \frac{v_\infty}{1 - \hat{k} \cdot v_\infty}$$

For which $w \in \mathbb{C}^3$

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"scattered states escape from Fock space"

We showed

1. $\mathcal{F}(\hat{k}) := \lim_{|k|\to 0} |k|\hat{F}(k,t)$ defines a conserved quantity,
2. the soft-photon theorem of the form
$\mathcal{F}_{\mathrm{sc},+}(\hat{k}) - \mathcal{F}_{\mathrm{sc},-}(\hat{k}) = -\left(\mathcal{F}_{v_{+\infty}}(\hat{k}) - \mathcal{F}_{v_{-\infty}}(\hat{k})\right).$

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