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On the construction of Feynman Parametrices for Normally Hyperbolic Operators

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(joint work with Alexander Strohmaier)

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Introduction

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- Normally hyperbolic operator: $\Box \in \mathcal{P}^2(\mathscr{M}, \mathscr{E})$ st $\sigma_\Box(\xi) = \mathrm{g}^\sharp(\xi, \xi)$. In local coordinates $(x^0, \ldots, x^{d-1})$ on globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ after trivialising the complex smooth vector bundle $\mathscr{E}$

$$\Box = \mathrm{g}^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} + A^\mu \frac{\partial}{\partial x^\mu} + B,$$

A^μ, B-matrix valued coefficients.

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- **Do Feynman parametrices exist for a NHOp on GHSTs?**

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- Feynman (1949) propagator: expectation value of time ordered (T) massive (m) scalar fields wrt Minkowski vacuum ω_0

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where E is defined locally and covariantly whereas R_ω is ω-dependant nonlocal smooth terms.

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Microlocal Definition of Feynman parametrix

A map $E : \mathcal{D}(\mathscr{M},\mathscr{E}) \xrightarrow{C} \mathcal{E}(\mathscr{M},\mathscr{E})$ st (Duistermaat & Hörmander, 1972)

$$\mathrm{WF}'(E) \subseteq \Delta_{\dot{T}^*\mathscr{M}^2} \bigcup \{(x,\xi;y,\eta) \in \dot{T}^*\mathscr{M}^2 | \mathrm{g}_x(\xi,\xi) = 0, \exists s \in \mathbb{R}_{\gtrless 0} : (x,\xi) = \Phi_s(y,\eta)\}.$$

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- Take home message of this talk

Theorem (Existence & uniqueness of Feynman parametrices)

Let $\mathscr{E} \to \mathscr{M}$ be a smooth complex vector bundle over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ and $\Box : \mathcal{E}(\mathscr{M}, \mathscr{E}) \to \mathcal{E}(\mathscr{M}, \mathscr{E})$ is a NHOp. Then there exist unique Feynman parametrices for $\Box$.

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- If $(x,\xi),(y,\eta) \notin \mathrm{WF}\,(\mathscr{P})$ but $(x,\xi;y,\eta) \in \mathrm{WF}'\,(L\mathscr{P}R)$ then $\exists$ $(z,\zeta) \in \mathrm{WF}\,(\mathscr{P})$ st $(x,\xi) \gtrsim (z,\zeta) \lesssim (y,\eta)$ on geodesic strip $\gamma(t)$.

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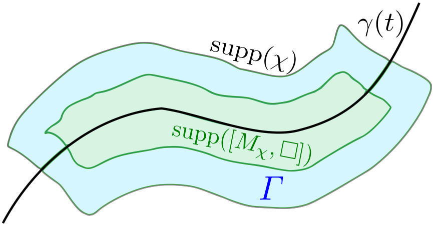
- Choose $\chi \in \mathcal{D}(\mathscr{M}\times\mathscr{M},\mathscr{E}\boxtimes\mathscr{E})$ st $\chi \equiv 1$ in a nbh. Γ of the proj. $\{(x,\xi;y,\eta) \in \dot{T}^*\mathscr{M}^2 | \mathrm{g}_x(\xi,\xi)=0, (x,\xi)=\Phi^{\gamma}_{s\gtrless 0}(y,\eta)\}$ on $\mathscr{M}\times\mathscr{M}$.



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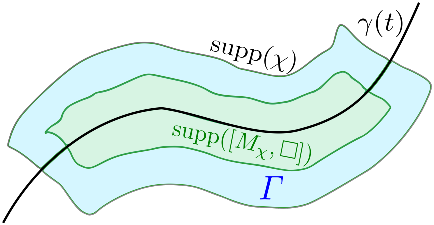
$$\mathrm{supp}([\Box, M_\chi]) \subset \mathrm{supp}(\chi) \setminus \Gamma,$$



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$$L[M_\chi,\Box]R$$
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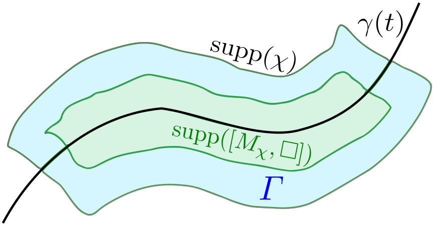

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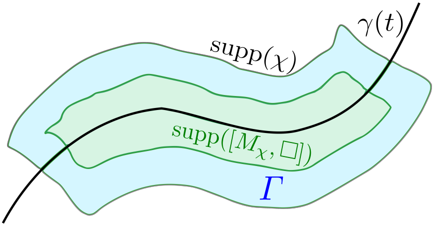
- If $(x,\xi),(y,\eta) \notin \mathrm{WF}(\mathscr{P})$ but $(x,\xi;y,\eta) \in \mathrm{WF}'(L\mathscr{P}R)$ then $\exists$ $(z,\zeta) \in \mathrm{WF}(\mathscr{P})$ st $(x,\xi) \gtrless (z,\zeta) \lessgtr (y,\eta)$ on geodesic strip $\gamma(t)$.
- Choose $\chi \in \mathcal{D}(\mathscr{M}\times\mathscr{M},\mathscr{E}\boxtimes\mathscr{E})$ st $\chi \equiv 1$ in a nbh. Γ of the proj. $\{(x,\xi;y,\eta) \in \dot{T}^*\mathscr{M}^2 | \mathrm{g}_x(\xi,\xi)=0, (x,\xi)=\Phi^{\gamma}_{s\gtrless 0}(y,\eta)\}$ on $\mathscr{M}\times\mathscr{M}$.

$$\mathrm{supp}([\Box, M_\chi]) \subset \mathrm{supp}(\chi) \setminus \Gamma,$$

$$L[M_\chi,\Box]R$$
$$= LM_\chi\Box R - L\Box M_\chi R,$$

$$\Rightarrow \mathrm{singsupp}(L[\Box, M_\chi]R) \neq \Gamma$$



Uniqueness of Feynman Parametrix

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- *$L\Box R$ congruent both to L and to R modulo smoothing operators.*
- $\mathcal{E}'(\mathscr{M},\mathscr{E}) \xrightarrow{R} \mathcal{D}'(\mathscr{M},\mathscr{E}) \xrightarrow{\Box} \mathcal{D}'(\mathscr{M},\mathscr{E}) \xrightarrow{L} \mathcal{D}'(\mathscr{M},\mathscr{E})$ not defined.
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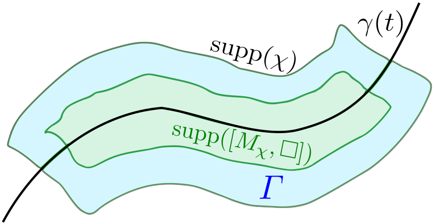

Hence $LM_\chi - M_\chi R$ does not contain any point over Γ and therefore $L-R$ is a smoothing op. as Γ is arbitrary.

Construction of Feynman Parametrix (sketch)

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4. Feynman parametrices for $\Box$:

$$\Psi_{\text{ell}}^{+1} E \tag{5}$$

Fourier integral operators (FIOs)

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1. A FIO is a linear operator (Hörmander, 1971; Lax, 1957)

$$A : \mathcal{D}(\mathscr{N}, \sqrt{\Omega\mathscr{N}} \otimes \mathscr{F}) \to \mathcal{D}'(\mathscr{M}, \sqrt{\Omega\mathscr{M}} \otimes \mathscr{E}), (Au)(x) := \int_{\mathscr{N}} \mathsf{A}(x, y)u(y)$$

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2. Locally a FIO can be written as a matrix whose elements are

$$\mathsf{A}(x) = (2\pi)^{(d+2n-2e)/4} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}S(x,\theta)} a(x, \theta) \mathrm{d}\theta \qquad (6)$$

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5. Principal symbol of $P \in \Psi^m(\mathscr{M}, \mathscr{E})$

   $\sigma_P : \Psi^m/\Psi^{m-1}(\mathscr{M}, \mathscr{E}) \to S^m/S^{m-1}(T^*\mathscr{M}, \operatorname{End}(\mathscr{E}))$.

Microlocalisation

- $(x, \xi) \in \dot{T}^*\mathscr{M}$ lightlike covector, $(0, \eta) := (0; \eta_0, 0, \dots, 0) \in \dot{T}^*\mathbb{R}^d$.

Microlocalisation

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$$\mathrm{CN}_{(x,\xi)} \xleftarrow{\varkappa} \mathrm{CN}_{(0,\eta)}$$

Figure 1: A schematic diagram of microlocalisation.

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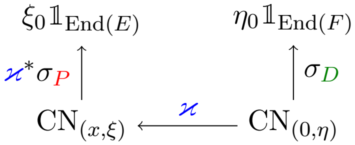

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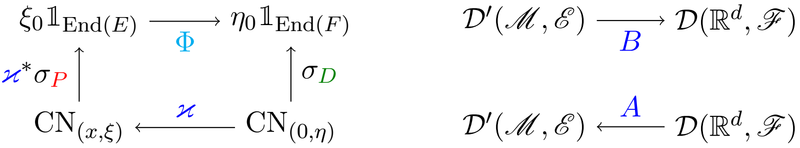

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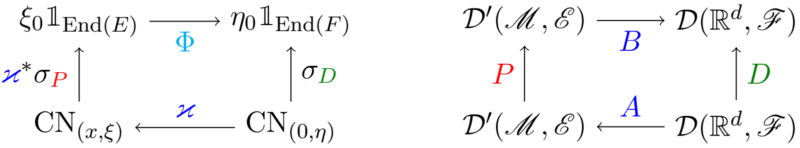
$$\begin{gathered} \mathrm{WF}'(A) \subset \Gamma_{(x,\xi;0,\eta)}, \quad \mathrm{WF}'(B) \subset \Gamma_{(0,\eta;x,\xi)}, \\ (x, \xi) \notin \mathrm{WF}(BA - I), \quad (0, \eta) \notin \mathrm{WF}(AB - I), \end{gathered} \tag{8a}$$

Microlocalisation



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Facts used in the Proof of Microlocalisation

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- Existence of parametrix for FIOs.

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Thank you for your attention!

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