High-energy bounds on Møller operators

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Joint work with D. Cadamuro (Leipzig) and G. Lechner (Cardiff), arXiv:1912.11092

Content of the talk

- For two selfadjoint operators H_0 and $H_1 = H_0 + V$, consider the Møller operators

$$\Omega_\pm := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_1} e^{-itH_0}.$$

- Heuristic expectation: At high energies (i.e., on subspaces for large spectral values of H_0), one has $\Omega_\pm \approx \mathbf{1}$.
- More precisely, we ask whether for some continuous function f,

$$\|(\Omega_\pm - \mathbf{1})f(H_0)\| < \infty.$$

- When does this happen? How fast can f grow?
- How to find “effective” criteria for this bound to hold?
- Motivating example from quantum physics: “quantum backflow”

Motivating example

A motivating example from quantum mechanics:

- Consider a free particle in one dimension, $\psi \in L^2(\mathbb{R})$
- Probability flux at point x, averaged with test function $g \geq 0$:
  - $J(g) = \int g(x)J(x) = \frac{1}{2}(Pg(X) + g(X)P)$
  - $\langle \psi, J(x)\psi \rangle = \frac{1}{2mi}\big(\overline{\psi(x)}\psi'(x) - \overline{\psi'(x)}\psi(x)\big)$.
- Spectral values of the probability flux:
  - $J(g)$ has spectrum in all $\mathbb{R}$.
  - Let E be the projector onto positive momentum, then $EJ(g)E$ has spectrum in some interval $[-\epsilon, \infty)$
  - Different from classical mechanics, it is not positive ("quantum backflow effect")
  - Instead, it is bounded below ("quantum inequality"; Eveson/Fewster/Verch '03).
  - This is reminiscent of "quantum energy inequalities" in QFT.

$EJ(g)E$ is bounded below.
What happens when scattering is present?

- Is $E\Omega_\pm^* J(g)\Omega_\pm E$ bounded below?

$$\|\Omega_\pm^* J(g)\Omega_\pm - J(g)\| \leq 2\|(\Omega_\pm - 1)^*(1+H)^{1/2}\| \, \|(1+H)^{-1/2}J(g)\|.$$

- Hence the "quantum inequality" is stable under scattering if $\|(\Omega_\pm - 1)^*(1+H)^{1/2}\| < \infty$.
- We showed this by a direct argument (B./C./L. 2017) for a wide class of potentials V.
- But is there a general principle underlying?

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Pushing Particles Forwards Might Make Them Go Backwards Because Quantum Physics Is Bonkers

Ryan F. Mandelbaum
7/18/17 6:08pm

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"We wanted to show this is a universal quantum mechanical effect," study author Daniela Cadamuro from the Technical University of Munich in Germany told Gizmodo. "In the presence or absence of a force, the particle will always have a probability to move backward, even if there is a positive momentum."

One of quantum mechanics' core tenets is that the smallest particles act like dots and flowing waves at the same time. That's demonstrated by a quintessential experiment: If you shoot

A more general setting – abstract scattering theory

Consider two selfadjoint operators H_0, H_1 on a Hilbert space $\mathcal{H}$, let P_j^{ac} project onto their space of absolutely continuous spectrum, and define the Møller operators

$$\Omega_\pm(H_1, H_0) := \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_1} e^{-itH_0} P_0^{\mathrm{ac}}.$$

(You may think $H_1 = H_0 + V$, but is that the right viewpoint?)

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Definition

Let H_0 and H_1 be self-adjoint operators such that $\Omega_\pm(H_1, H_0)$ exist; let $f \in C(\mathbb{R})$. Then (H_1, H_0) is called *f*-bounded if $(\Omega_\pm(H_1, H_0) - P_1^{\mathrm{ac}} P_0^{\mathrm{ac}}) f(H_0)$ is bounded.

Is this fulfilled in examples? Which f can be chosen?

Some a priori examples

How strong is the condition of f-boundedness?

Example A: all f-bounds

- Consider $H_0 = -i\partial_x$ on $\mathcal{H} = L^2(\mathbb{R})$,
  $H_1 = -i\partial_x + P_\xi$ where P_ξ projects onto some ξ with $\tilde{\xi}$ compactly supported.
- In that case, $H_1 = H_0$ on subspaces of large momenta, and $\Omega_\pm = 1$ there.
- Hence $(\Omega_\pm - \mathbf{1})f(H_0)$ is bounded no matter what $f \in C(\mathbb{R})$ we choose!

Example B: no f-bounds

- Take $H_0 = -i\partial_x$ on $\mathcal{H} = L^2(\mathbb{R})$, and $H_1 = -i\partial_x + v(x)$.

$$(\Omega_\pm \psi)(x) = w_\pm(x)\psi(x)\,, \quad w_\pm(x) := \exp i \int_x^{\pm\infty} v(y)\, dy.$$

- Let $(U(p)\psi)(x) = e^{ipx}\psi(x)$. Then $U(p)\Omega_\pm U(p)^* = \Omega_\pm$ and
  $U(p)f(H_0)U(p)^{-1} = f(H_0 - p)$.
- Hence if (H_1, H_0) was f-bounded, then $(\Omega_\pm - \mathbf{1})f(H_0 - p)$ is uniformly bounded in p.

Towards f-boundedness

How can we estimate $(\Omega_\pm - \mathbf{1})f(H_0)$ in the general case?

- We need a more explicit description of the Møller operator.
- For example, in terms of the resolvents $R_j(z) = (H_j - z\mathbf{1})^{-1}$:

$$\Omega_\pm(H_1, H_0) = \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\pi} \int R_1(\lambda \mp i\epsilon) R_0(\lambda \pm i\epsilon)\, d\lambda.$$

  Hence, at least formally,

$$\Omega_\pm(H_1, H_0) - \mathbf{1} = \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\pi} \int \Big(R_1(\lambda \mp i\epsilon) - R_0(\lambda \mp i\epsilon) \Big) R_0(\lambda \pm i\epsilon)\, d\lambda.$$

- To really obtain estimates here, we need more information about the limit values of the resolvents at the real axis.
- This is available in the smooth method of scattering theory.

Smooth method

Smooth method of scattering theory:

- Mostly applicable to differential operators and their perturbations.
- Here the boundary values $\lim_{\epsilon\downarrow 0} R(\lambda \pm i\epsilon)$ are taken seriously ("limiting absorption principle").
- Of course the limit does not exist in operator norm.
  But consider the following well-known example:
  - $H_0 = -\partial_x^2$ on $\mathcal{H} = L^2(\mathbb{R})$
  - Resolvent $R_0(z)$ has integral kernel

$$K(x, y; z) = \frac{i}{2\sqrt{z}} \exp\left(i\sqrt{z}|x-y|\right).$$

  - If $\alpha > \frac{1}{2}$, then $(1+x^2)^{-\alpha/2} K(x,y;z)(1+y^2)^{-\alpha/2}$ converges to a "good" (Hilbert-Schmidt) kernel as $\operatorname{Im} z \to 0+$.
- That suggests the following framework.

Setting of the smooth method

- Consider a Gelfand triple $\mathcal{X} \subset \mathcal{H} \subset \mathcal{X}^*$.
  - $\mathcal{X}$ a Banach space, $\mathcal{X}^*$ its conjugate dual
  - Scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathcal{H}$ yields $\mathcal{H} \subset \mathcal{X}^*$ via $\varphi \mapsto \langle\,\cdot\,,\varphi\rangle$.
  - Embeddings assumed continuous and dense.
- H_0 is called $\mathcal{X}$-smooth if

$$R_0(\lambda \pm i0) := \lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon)$$

  exist in $\mathfrak{B}(\mathcal{X},\mathcal{X}^*)$ for $\lambda \in U$, where U is an open set of full measure, and the extended R_0 is locally Hölder continuous.
- If $R_0(z) \in \mathrm{FA}(\mathcal{X},\mathcal{X}^*)$ for $\mathrm{Im}\, z \neq 0$, and $V \in \mathfrak{B}(\mathcal{X}^*,\mathcal{X})$, then $H_1 := H_0 + V$ is also $\mathcal{X}$-smooth (but U might change).
- If H_0, H_1 are both $\mathcal{X}$-smooth, and $V := H_1 - H_0 \in \Gamma_2(\mathcal{X}^*,\mathcal{X})$ then the Møller operators $\Omega_\pm(H_1, H_0)$ exist.

High-energy behaviour in the smooth method

So what about f-boundedness and the smooth method?

High-energy behaviour in the smooth method

So what about *f*-boundedness and the smooth method?

Definition

We say that an $\mathcal{X}$-smooth operator H is of high-energy order β if there exist $\hat{\lambda}, b > 0$ such that

$$\|R(\lambda \pm i0)\|_{\mathcal{X},\mathcal{X}^*} \leq b|\lambda|^{-\beta} \quad \text{for all } \lambda \in U, \ |\lambda| \geq \hat{\lambda}.$$

High-energy behaviour in the smooth method

So what about f-boundedness and the smooth method?

Definition

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Proposition

Let H_0, H_1 be $\mathcal{X}$-smooth, and let $H_1 - H_0 \in \mathfrak{B}(\mathcal{X}^*, \mathcal{X})$.
Then H_0 is of high-energy order β if and only if H_1 is.

Main result

"High-energy order β" implies "f_β-boundedness".

- Here $f_\beta(\lambda) = (1+\lambda^2)^{\beta/2}$, $\beta \in (0,1)$.

Theorem

Let H_0, H_1 be selfadjoint and $H_1 - H_0 \in \Gamma_2(\mathcal{X}^*, \mathcal{X})$.
If H_0, H_1 are $\mathcal{X}$-smooth and of high-energy order $\beta \in (0,1)$,
then H_0 and H_1 are mutually f_β-bounded.

- This is "symmetric" in H_1, H_0.
- It is "effective" since smoothness and high-energy order need to be shown for one of H_0, H_1 only (see earlier).

Example: Perturbed polyharmonic operator

- Let us consider the following example:
  - $\mathcal{H} = L^2(\mathbb{R}^n, dx)$
  - $\mathcal{X} = L^2(\mathbb{R}^n, (1+|x|^2)^\alpha dx)$, where $\alpha > \frac{1}{2}$.
  - $H_0 = (-\Delta)^{\ell/2}$ with some $\ell \in (1, \infty)$
  - $H_1 = H_0 + V$ where V is a multiplication operator, $\|V\|_{\mathcal{X}^*,\mathcal{X}} < \infty$.
- One shows that H_0 is $\mathcal{X}$-smooth and of high-energy order β for $0 < \beta \leq 1 - \frac{1}{\ell}$.
- Also, $R_0(z)$ is compact in $\mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ for $\operatorname{Im} z \neq 0$.
- Therefore, H_0 and H_1 are mutually f_β-bounded for all $0 < \beta \leq 1 - \frac{1}{\ell}$.
- The bound on β is strict in general.
  (Can construct counterexample for $n = 1$, $\ell = 2$, $\beta > \frac{1}{2}$.)

Conclusions

- We have investigated high-energy bounds on Møller operators in an abstract framework.
- These now allow to show stability of “quantum inequalities” under scattering.
- Concrete examples include $(-\Delta)^{\ell/2} + v(x)$
  (in particular, Schrödinger in any dimension).
- We can also handle inner degrees of freedom:
  - Take $\mathcal{H} = \mathcal{H}_{\mathrm{tr}} \otimes \mathcal{H}_{\mathrm{inner}}$, $H_0 = H_{\mathrm{tr}} \otimes 1 + 1 \otimes H_{\mathrm{inner}}$,
    $H_1 = H_0 + v(x)$ where v is $\mathfrak{B}(\mathcal{H}_{\mathrm{inner}})$-valued.
  - Under some conditions, the high-energy order of H_{tr} transfers to H_0.
  - Easy if H_{inner} is finite-dimensional.
  - Intricate (but possible) if H_{inner} has discrete spectrum.
- Apart from the smooth method, we also get results in the trace-class method (not discussed here).
- On the theoretical side, one can make f-boundedness into an equivalence relation.
- Extensions of these results?