Stability of anyonic superselection sectors

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Topological phases

Quantum phase outside of Landau theory

- ground space degeneracy
- long range entanglement
- anyonic excitations
- modular tensor category / TQFT
Modular tensor category

Describes all properties of the anyons, e.g. fusion, braiding, charge conjugation, ...

Irreducible objects $\rho_i \Leftrightarrow$ anyons
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How to get the modular tensor category?
Modular tensor category

Describes all properties of the anyons, e.g. fusion, braiding, charge conjugation, …

Irreducible objects $\rho_i \Leftrightarrow$ anyons

How to get the modular tensor category? Is this stable?
Our approach

- take an **operator algebraic approach**
- ...inspired by **algebraic quantum field theory**
- useful to study **structural questions**
- but also **concrete models**
- can make use of **powerful mathematics**
Quantum phases
Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathcal{A}(\Lambda) \cong \bigotimes_{x \in \Lambda} M_d(\mathbb{C})$

- quasilocal algebra $\mathcal{A} := \bigcup \mathcal{A}(\Lambda)$

- local Hamiltonians $H_\Lambda$ describing dynamics

- gives time evolution $\alpha_t$ & ground states

- if $\omega$ a ground state, Hamiltonian $H_\omega$ in GNS repn.
Quantum phases of ground states

Two ground states $\omega_0$ and $\omega_1$ are said to be in the same phase if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that $\omega_s$ is a ground state of $H(s)$.

(Chen, Gu, Wen, Phys. Rev. B 82, 2010)
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Alternative definition: $\omega_0$ can be transformed into $\omega_1$ with a finite depth local quantum circuit.
Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_{\Lambda} + \Phi(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

Superselection sectors
Example: toric code
Example: toric code

$\omega_0 \circ \rho$ is a single excitation state

$$\rho(A) := \lim_{n \to \infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes observables in presence of background charge
Localised and transportable morphisms

The endomorphism $\rho$ has the following properties:

- **localised**: $\rho(A) = A \quad \forall A \in \mathcal{A}(\Lambda^c)$

- **transportable**: for $\Lambda'$ there exists $\sigma$ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these properties (à la Doplicher-Haag-Roberts)

Doplicher, Haag, Roberts, Fredenhagen, Rehren, Schroer, Fröhlich, Gabbiani, …
Definition

A superselection sector is an equivalence class of representations $\pi$ such that

$$\pi|_{\mathcal{A}(\Lambda^c)} \cong \pi_0|_{\mathcal{A}(\Lambda^c)}$$

for all cones $\Lambda$. 
Theorem (Fiedler, PN)

Let $G$ be a finite abelian group and consider Kitaev’s quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\text{Rep} \ D(G)$.

General models

We can obtain a braided tensor category under general conditions:

- **Haag duality:** $\pi_0(\mathcal{A}(\Lambda))'' = \pi_0(\mathcal{A}(\Lambda^c))'$

- **split property:** $\pi_0(\mathcal{A}(\Lambda_1))'' \subset \mathcal{N} \subset \pi_0(\mathcal{A}(\Lambda_2))''$

- technical property related to direct sums

No reference to Hamiltonian!
Theorem

Let $\Lambda$ be a cone and suppose that $\omega_0$ is a pure state equivalent to $\omega_\Lambda \otimes \omega_{\Lambda^c}$. Then the corresponding GNS representation $\pi_0$ has no non-trivial super selection sectors.

PN, Ogata, work in progress
Stability
Stability

How much of the structure is invariant?

Does the gap stay open under small perturbations?

Is the superselection structure preserved?
Kato, PN, arXiv:1810.02376
Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

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This is not enough to conclude stability of the superselection structure!
The superselection criterion is defined on the C*-algebraic level...

... but full analysis requires von Neumann algebras (also, split property, Haag duality for $\pi_0$).

For example, **intertwiners** $V \in \pi_0(\mathcal{A}(\Lambda))''$

Not clear if/how $\alpha_S$ extends
Almost localised endomorphisms
No strict localisation
Almost localised endomorphisms

An endomorphism $\rho$ of $\mathcal{A}$ is called \textit{almost localised} in a cone $\Lambda_\alpha$ if

$$\sup_{A \in \mathcal{A}(\Lambda_{\alpha+\epsilon}^c + n)} \frac{\|\rho(A) - A\|}{\|A\|} \leq f_\epsilon(n)$$

where $f_\epsilon(n)$ is a non-increasing family of absolutely continuous functions which decay faster than any polynomial in $n$. 
Define a semigroup $\Delta$ of endomorphisms that are

- **almost localised** in cones

- **transportable:** for $\Lambda'$ there exists $\sigma$ almost localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

- **intertwiners** $(\rho, \sigma)_{\pi_0} := \{ T : T\pi_0(\rho(A)) = \pi_0(\sigma(A))T \}$

Can we do sector analysis again?
Stability of Kitaev’s quantum double
Almost localised endomorphisms

Follow strategy of Buchholz et al.: asymptopia

Most tricky part: define tensor structure

\[(\rho \otimes \sigma)(A) := \rho \circ \sigma(A)\]

\[(\rho, \sigma)_{\pi} := \{T : T\pi(\rho(A)) = \pi(\sigma(A))T\}\]

\(T\) in general not in \(\mathcal{A}\)! How to define \(S \otimes T\)?

Intuitively: \(S \otimes T = S\rho(T)\)

**Haag duality** is not available!

Asymptotically inner

For general endomorphisms, there are \( \{U_n\} \subset \mathcal{B}(\mathcal{H}) \)

\[
\pi_0(\rho(A)) = \lim_{n \to \infty} U_n \pi_0(A) U_n^*
\]

Sequences are not unique, look at such collections:

\[
\rho(A) = \lim_n U_n A U_n^*, \quad \rho'(A) = \lim_n V_n A V_n^*
\]

and \( R \in (\rho, \rho')_{\pi_0}, \quad R' \in (\sigma, \sigma')_{\pi_0} \)

\[
\lim_{m,n \to \infty} \|[V_n RU_m^*, R']\| = 0
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Asymptotically inner

For general endomorphisms, there are \( \{U_n\} \subset \mathcal{B}(\mathcal{H}) \)

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Enough to define fusion

\[
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and \( R \in (\rho, \rho')_{\pi_0}, \quad R' \in (\sigma, \sigma')_{\pi_0} \)

\[
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\]

Asymptopia

Follow strategy of Buchholz et al.: (bi-)asymptopia

Using approximate localisation we can get control over the support of \( \{ U_n \} \)

Use this to construct bi-asymptopia and obtain braided tensor category

Lieb-Robinson for cones

Quasi-local evolution send observables localised in cones to almost localised observables:

Let $X$ be a cone and $Y$ a cone with a slightly larger opening angle. Then with $A \in \mathcal{A}(X), B \in \mathcal{A}(Y^c + n)$

$$\|[\tau_t(A), B]\| \propto \|A\|\|B\|p(d(X, Y + n))e^{-vt-d(X,Y+n)}$$

Schmitz, Diplomarbeit Albert-Ludwigs-Universität Freiburg (1983)
An energy criterion

How are these models related?

Def: write $\mathcal{S}(s)$ for the set of weak-$\ast$ limits of all states which are mixtures of states with energy $< 5$. The category $\Delta^{qd}(s)$ consists of all endomorphisms that are:

$\triangleright$ almost localised and transportable (wrt. $\omega_0 \circ \alpha_s$)

$\triangleright$ $\omega_0 \circ \alpha_s \circ \rho \cong \omega, \quad \omega \in \mathcal{S}(s)$
Putting it all together

- (bi-)asymptopia give braided tensor category $\Delta^{qd}(s)$
- LR bounds give localisation in cones
- can use this to prove $\Delta^{qd}(s) \cong \alpha_s^{-1} \cdot \Delta^{qd}(0) \cdot \alpha_s$
- unperturbed model is well understood
- need energy criterion
Theorem

Let $G$ be a finite abelian group and consider the perturbed Kitaev’s quantum double model. Then for each $s$ in the unit interval, the category $\Delta_{qd}^s$ category is braided tensor equivalent to $\text{Rep} D(G)$.

Cha, PN, Nachtergaele, arXiv:1804.03203
Open problems

Non-abelian examples

Energy criterion

When do we get sectors?