Long-range entanglement and the split property

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Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

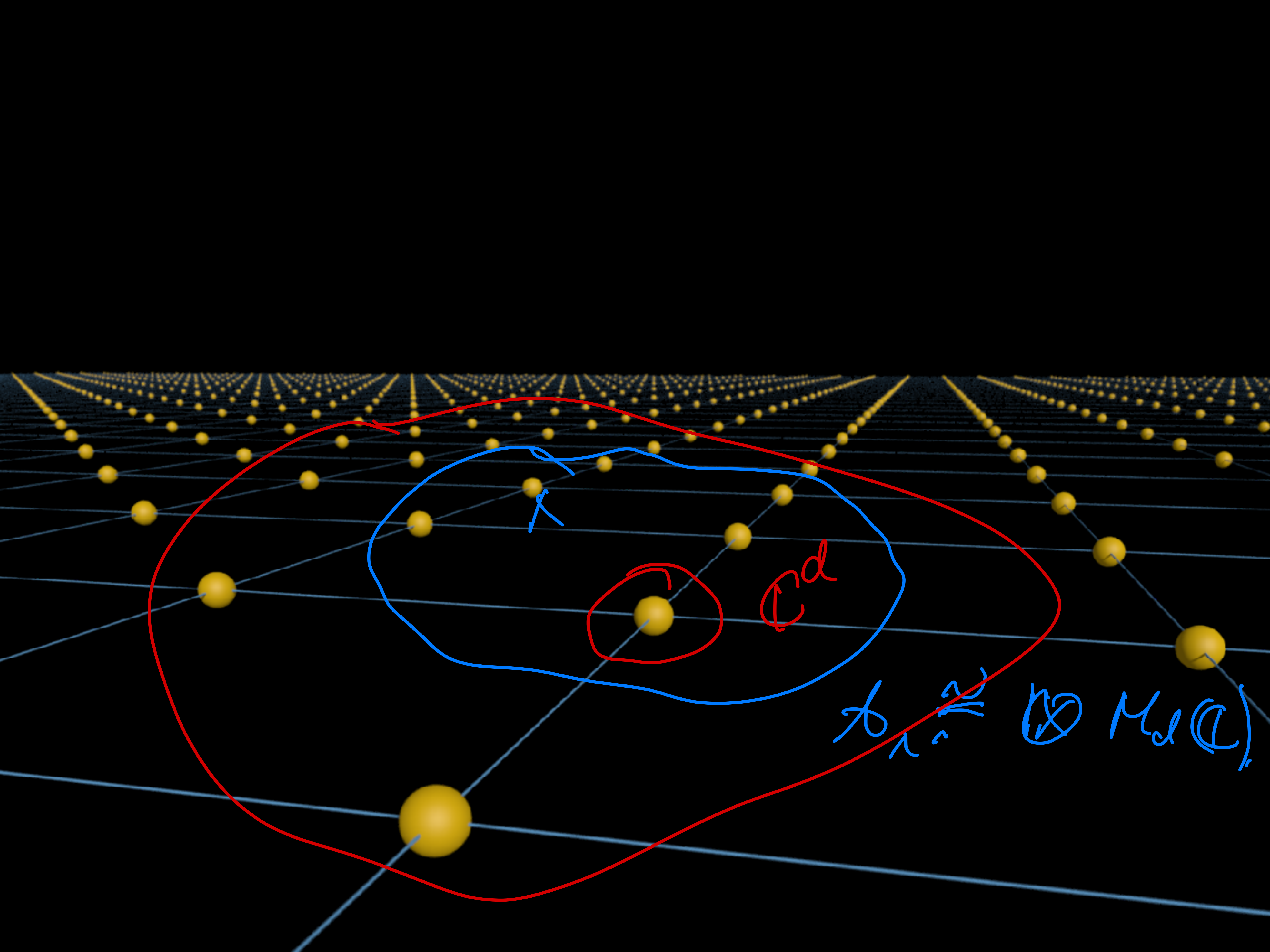
Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

- What are interesting phases?
- Can we find invariants?

Folklore

Topological order (and in particular anyonic excitations) are due to long range entanglement

Quantum phases


$\mathbb{C}^d$
$\mathcal{A}_\Lambda \cong \bigotimes M_d(\mathbb{C})$

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

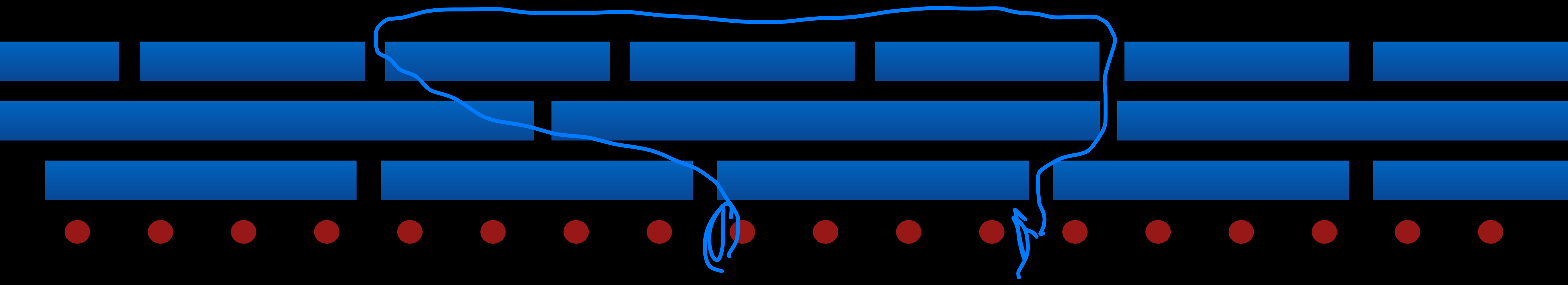
- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution α_t & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.

Quantum phases of ground states

Two ground states ω_0 and ω_1 are said to be *in the same phase* if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that ω_s is a ground state of $H(s)$.

(Chen, Gu, Wen, *Phys. Rev. B* **82**, 2010)

Alternative definition: ω_0 can be transformed into ω_1 with a *finite depth local quantum circuit.*



Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi_\Lambda(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

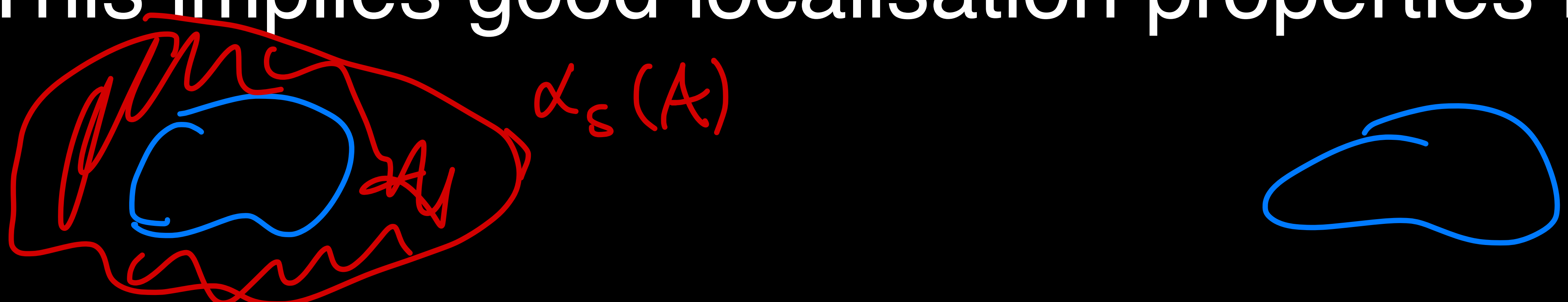
Commun. Math. Phys. **309** (2012)

Quasi-locality

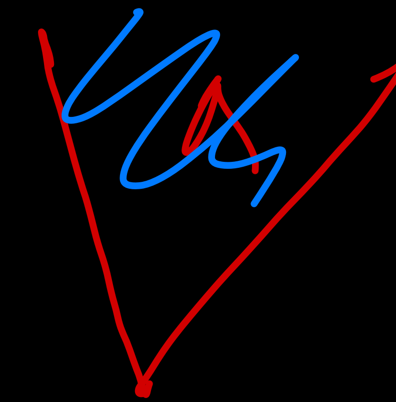
The main feature about the family of automorphisms α_s is that they are **quasi-local**, i.e. satisfy a **Lieb-Robinson** type of bound:

$$\|[\alpha_s(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F}\left(e^{C_\Phi} - 1\right)|X|\,G_F(d(X,Y))$$

This implies good localisation properties for α!

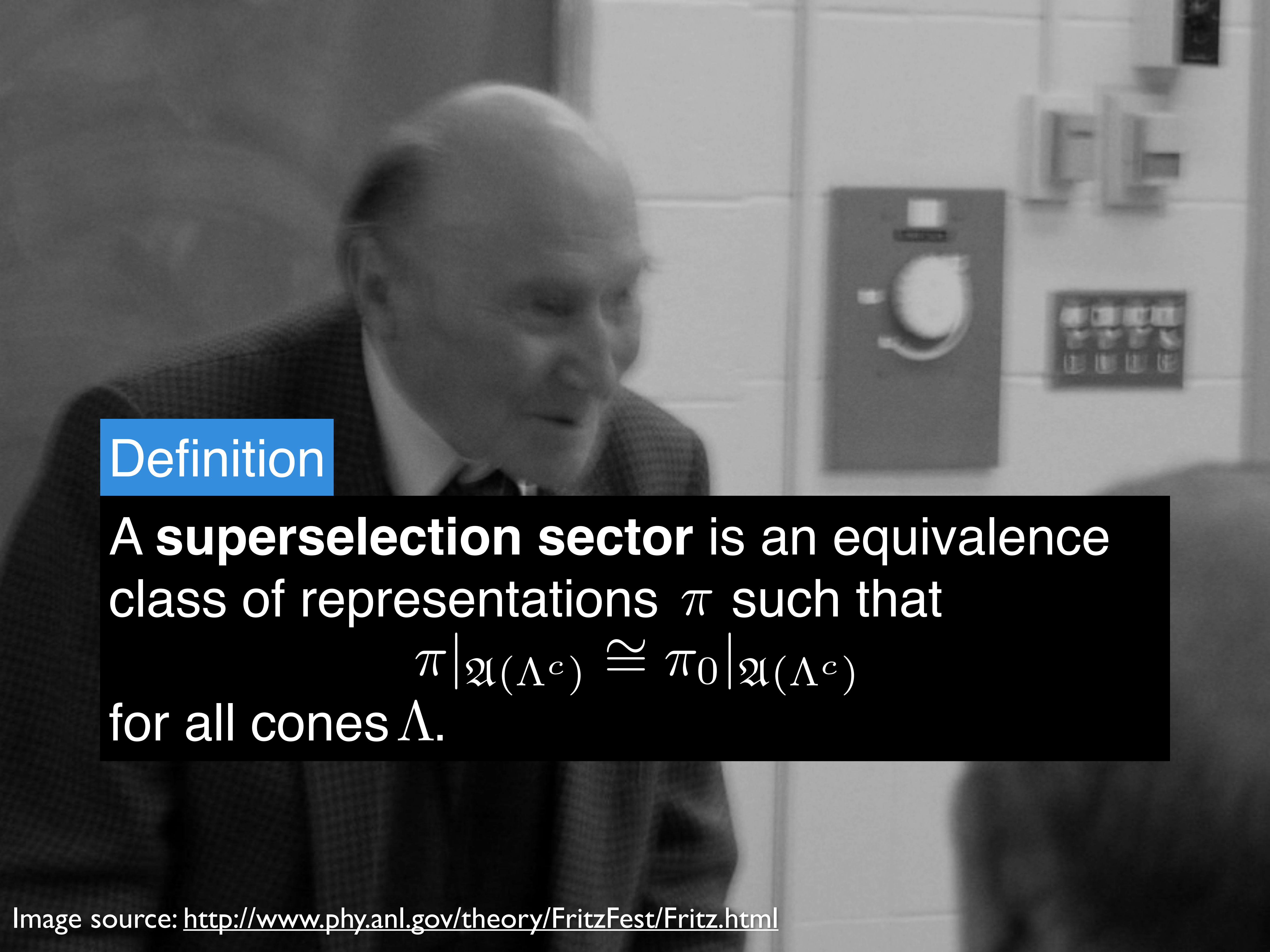


Long-range entanglement



- Bipartite system $\mathfrak{A}_\Lambda \otimes \mathfrak{A}_{\Lambda^c}$
- Product states $\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}$ have only classical correlations
- LRE: $\omega \circ \alpha$ is not quasi-equivalent to a product state for any quasi-local automorphism
- In 1D, gapped ground states are not LRE, in 2D this can be different!

Sector theory



Definition

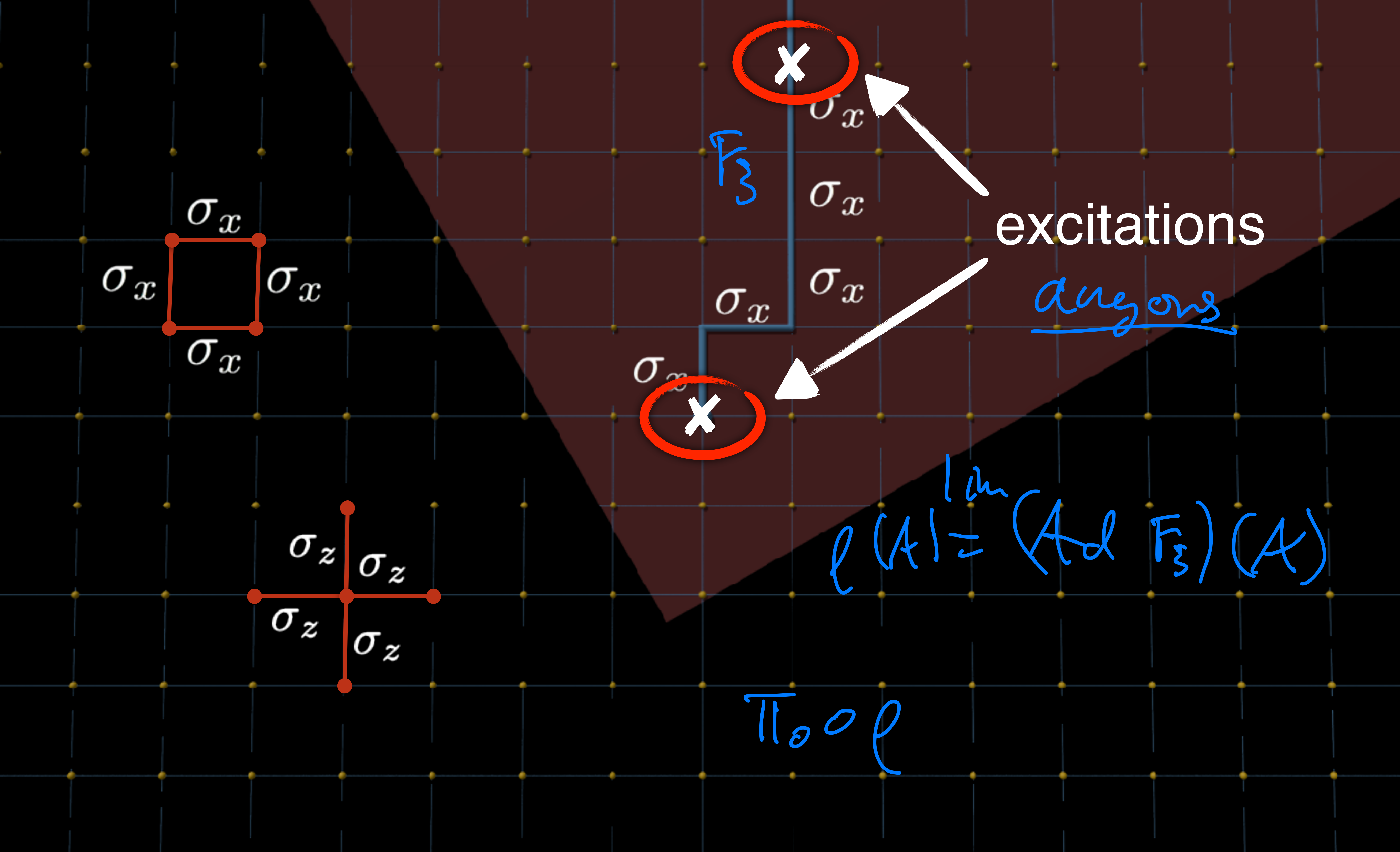
A **superselection sector** is an equivalence class of representations π such that

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

for all cones Λ.

Image source: http://www.phy.anl.gov/theory/FritzFest/Fritz.html

Example: toric code



Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\mathrm{Rep}\, D(G)$.

Rev. Math. Phys. **23** (2011)
J. Math. Phys. **54** (2013)
Rev. Math. Phys. **27** (2015)

Theorem

Let G be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each s in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\mathrm{Rep}\, D(G)$.

Cha, PN, Nachtergaele, *Commun. Math. Phys.* **373** (2020)

Long range entanglement

A new superselection criterion

We can relax the superselection criterion:

$$\pi \,|\, \mathfrak{A}_{\Lambda^c} \sim_{qe} \pi_\omega \,|\, \mathfrak{A}_{\Lambda^c}$$

That is, *quasi* instead of *unitary* equivalence

Remark: can be constructed naturally in non-abelian theories!

Szlachányi &Vecsernyés, CMP **156**, 1993

Theorem

Let ω be a pure state such that its GNS representation is quasi-equivalent to $\pi_\Lambda \otimes \pi_{\Lambda^c}$ for some cone Λ. Then the corresponding superselection structure is trivial.

PN & Y. Ogata, arXiv:2102.07707

Split property

A key ingredient in the proof is that the *split property* holds:

→ Type I factor.

$$\pi_\omega(\mathfrak{A}_\Lambda)'' \subset \mathcal{N} \subset \pi_\omega(\mathfrak{A}_{\Lambda^c})'$$

Equivalently, for ω pure:

$$\omega \sim_{qe} \omega_\Lambda \otimes \omega_{\Lambda^c}$$

Since $\mathcal{N}$ is a Type I factor: $\mathcal{H}_\omega \simeq \mathcal{H}_{\omega_\Lambda} \otimes \mathcal{H}_{\omega_{\Lambda^c}}$

From selection criterion $\pi \,|\, \mathfrak{A}_{\Lambda^c} \sim_{qe} \pi_\omega \,|\, \mathfrak{A}_{\Lambda^c}$ we get:

$\Rightarrow \qquad (I \otimes V)\pi(AB)(I \otimes V)^* = \pi_\Lambda(A) \otimes \pi_{\Lambda^c}(B) \otimes I_{\mathcal{K}}$

The trivial phase

This shows that Kitaev's toric code cannot satisfy the split property

Can it still be in the same phase?

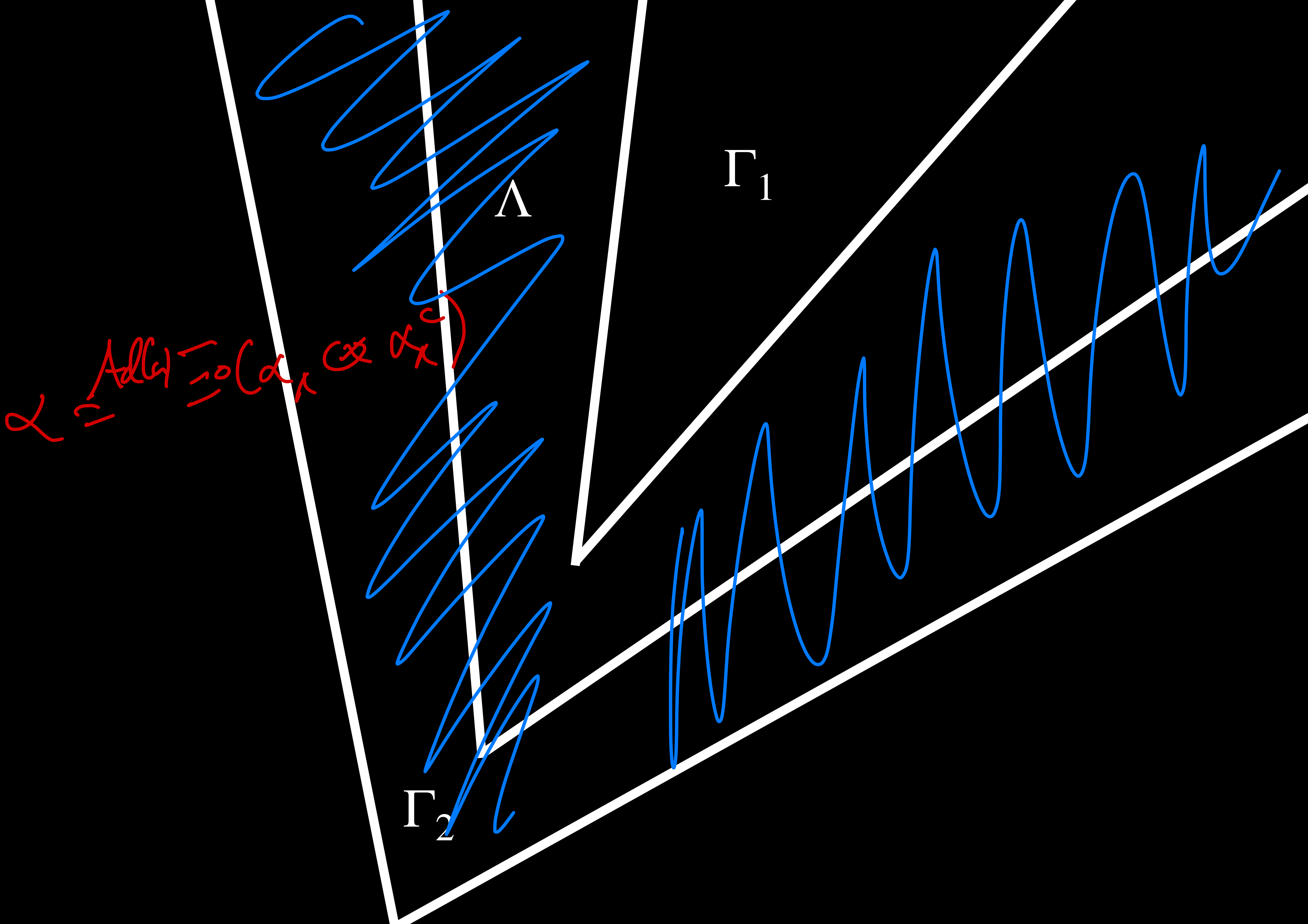
Definition

Consider an inclusion $\Gamma_1 \subset \Lambda \subset \Gamma_2$ of cones. Then $\alpha \in \mathrm{Aut}(\mathfrak{A})$ is called *quasi-factorisable* if:

$$\alpha = \mathrm{Ad}(u) \circ \Xi \circ (\alpha_\Lambda \otimes \alpha_{\Lambda^c})$$

for some unitary *u* and "local" automorphisms (see picture).

PN & Y. Ogata, arXiv:2102.07707


Λ
Γ1
Γ2
α = Ad(u) ∘ (α_Λ ⊗ α_Λ^c)

Theorem

Let π_0 be a representation and α quasi-factorisable for every cone. Then if π satisfies the selection criterion for π_0, then so does $\pi \circ \alpha$ for $\pi_0 \circ \alpha$.

Corollary

States in the trivial phase have trivial superselection structure.

PN & Y. Ogata, arXiv:2102.07707

The technical ingredients

Proof sketch

- Split property/Type I factor give factorisation of Hilbert space wrt. cone
- Superselection criterion implies similar factorisation for representation π (up to amplification)
- Quasi-factorisability preserves locality properties in suitable sense
- Then show that suitable α exist!

Theorem 3.1. *Let (Γ, d) be a countable ν-regular metric space with constant κ as in (2.1). Let F be an F-function on (Γ, d) such that the function G_F defined by (2.19) satisfies (2.34) for some $\alpha \in (0,1)$. Suppose that there is an F-function $\tilde{F}$ satisfying (2.35) for this F. Let $\mathcal{A}_\Gamma$ be a quantum spin system given by (2.3) and (2.4).*

Let $\Phi \in \mathcal{B}_F([0,1])$ be a path of interactions satisfying $\Phi_1 \in \mathcal{B}_F([0,1])$. (Recall from definition (2.22) that this means that $X \mapsto |X|\Phi(X;t)$ is in $\mathcal{B}_F([0,1])$). Let

$$\Gamma_1' \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma_2' \subset \Gamma. \tag{3.1}$$

For $m \in \mathbb{N} \cup \{0\}$, $x, y \in \Gamma$, set

$$f(m,x,y) := \sum_{X \ni x,y, d\left(\left(\Gamma_2' \setminus \Gamma_1'\right)^c, X\right) \le m} |X| \sup_{t \in [0,1]} \|\Phi(X,t)\| \,. \tag{3.2}$$

We assume that

$$\left(\sum_{x \in \Gamma_1} \sum_{y \in \Gamma_2^c} + \sum_{x \in \Gamma_2 \setminus \Gamma_1} \sum_{y \in (\Gamma_2 \setminus \Gamma_1)^c} \right) \sum_{m=0}^{\infty} G_F(m) f(m,x,y) < \infty \tag{3.3}$$

Define $\Phi^{(0)} \in \mathcal{B}_F([0,1])$ by

$$\Phi^{(0)}(X;t) := \begin{cases} \Phi(X;t), & \text{if } X \subset \Gamma_1 \text{ or } X \subset \Gamma_2 \setminus \Gamma_1 \text{ or } X \subset \Gamma_2^c \\ 0, & \text{otherwise} \end{cases}, \tag{3.4}$$

for each $X \in \mathcal{P}_0(\Gamma)$, $t \in [0,1]$. Then there is an automorphism $\beta_{\Gamma_2' \setminus \Gamma_1'}$ on $\mathcal{A}_{\Gamma_2' \setminus \Gamma_1'}$ and a unitary $u \in \mathcal{A}_\Gamma$ such that

$$\tau_{1,0}^{\Phi} = \mathrm{Ad}(u) \circ \tau_{1,0}^{\Phi^{(0)}} \circ \left(\tilde{\beta}_{\Gamma_2' \setminus \Gamma_1'} \right). \tag{3.5}$$

Conditions can be checked in relevant examples!