Anyons and long-range entanglement

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Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

- What are interesting phases?
- Can we find invariants?

Topological order

short-range entangled

Example: 1D gapped phases

Topological features from protected symmetries

long-range entangled

“intrinsic topological order”

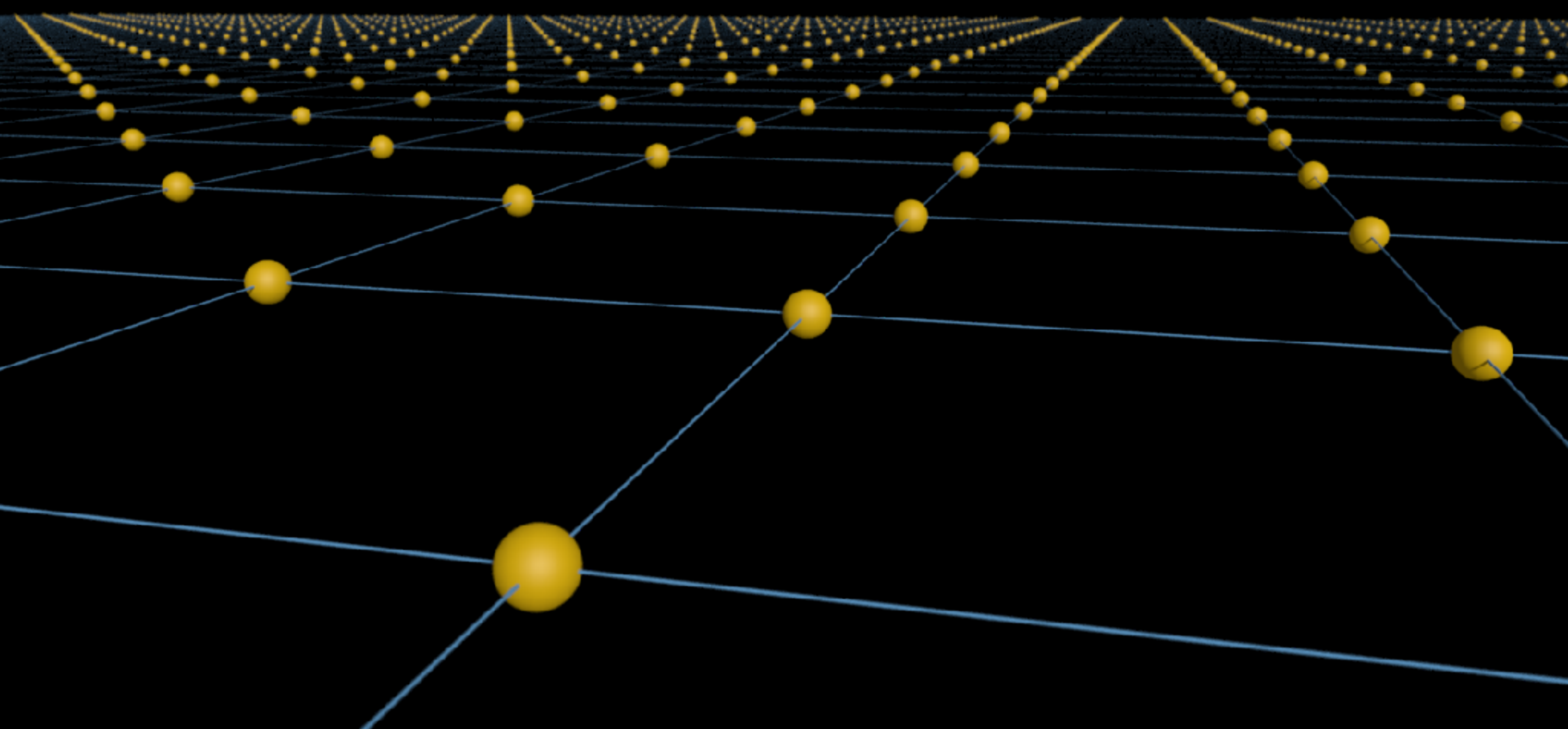
Example: toric code

Ground space degeneracy

Anyonic quasiparticles

Modular tensor category

Quantum phases in an operator algebraic framework



Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution $\alpha_t \in \mathrm{Aut}(\mathfrak{A})$ & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.

Finite depth quantum circuit

Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

Commun. Math. Phys. **309** (2012)

Quasi-locality

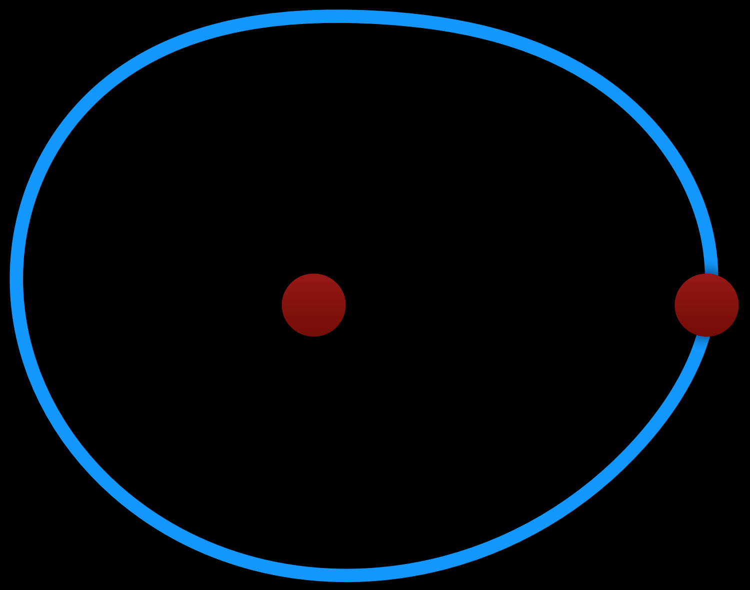
The main feature about the family of automorphisms α_s is that they are **quasi-local**, i.e. satisfy a **Lieb-Robinson** type of bound:

$$\|[\alpha(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F}\left(e^{C_\Phi} - 1\right)|X|\, G_F(d(X, Y))$$

This implies good localisation properties for α!

Anyons

Anyons in 2D



not contractible!

In quantum mechanics (abelian case):

$$\psi \to e^{i\theta}\psi$$

Leinaas & Myrheim (1977), Wilczek (1982)

Anyons and modular tensor categories

anyon types ⇔ irreducible objects

fusion of charges ⇔ $\rho_i \otimes \rho_j = \sum_k N_{ij}^k \rho_k$

conjugate charge ⇔ duals/conjugates

exchanging anyons ⇔ braiding

detect anyons through braiding ⇔ modularity

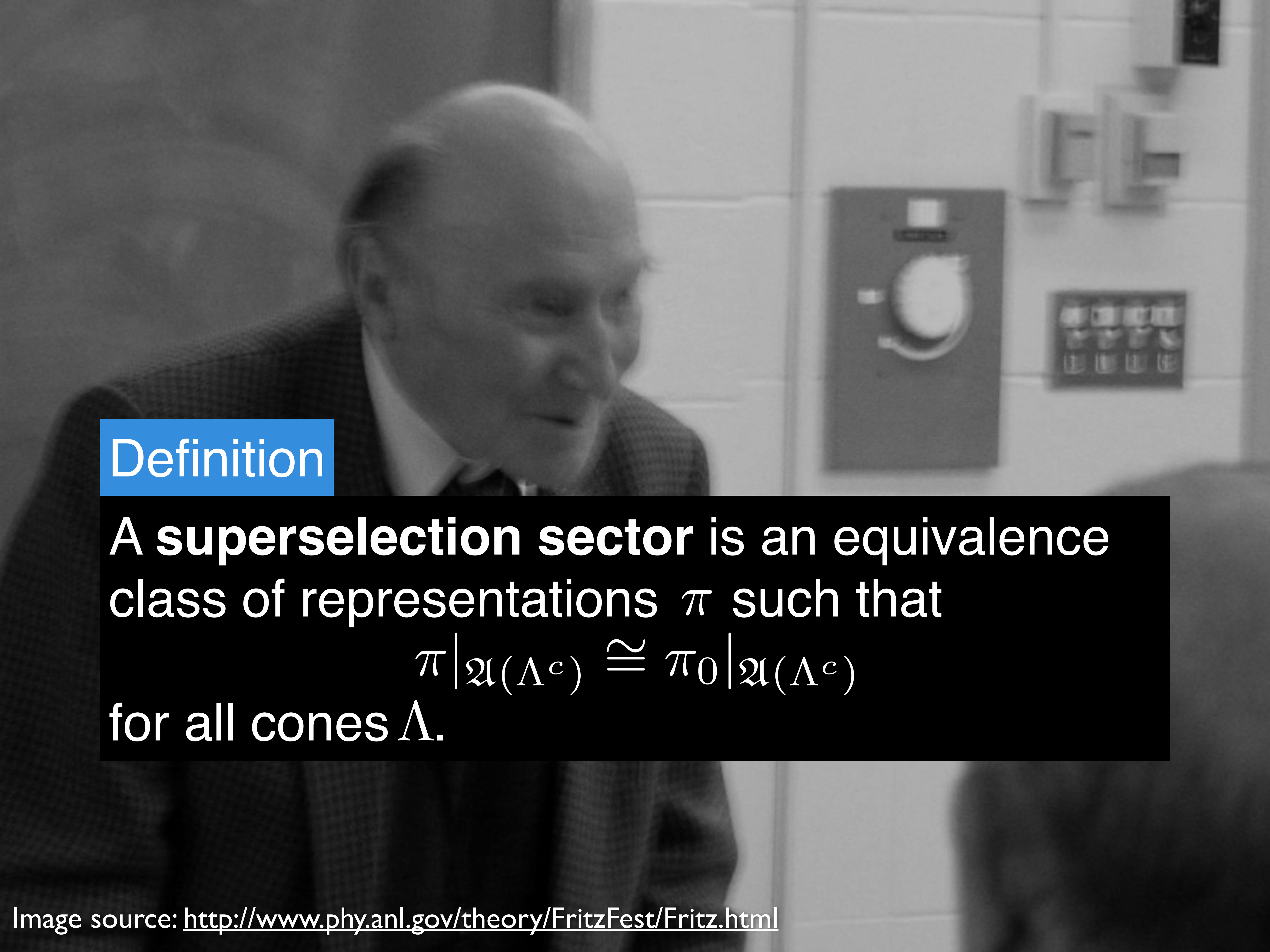
Problems:

How to get the MTC?

Is this an invariant?

LRE and trivial phases

Doplicher-Haag-Roberts sector theory



Definition

A **superselection sector** is an equivalence class of representations π such that

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

for all cones Λ.

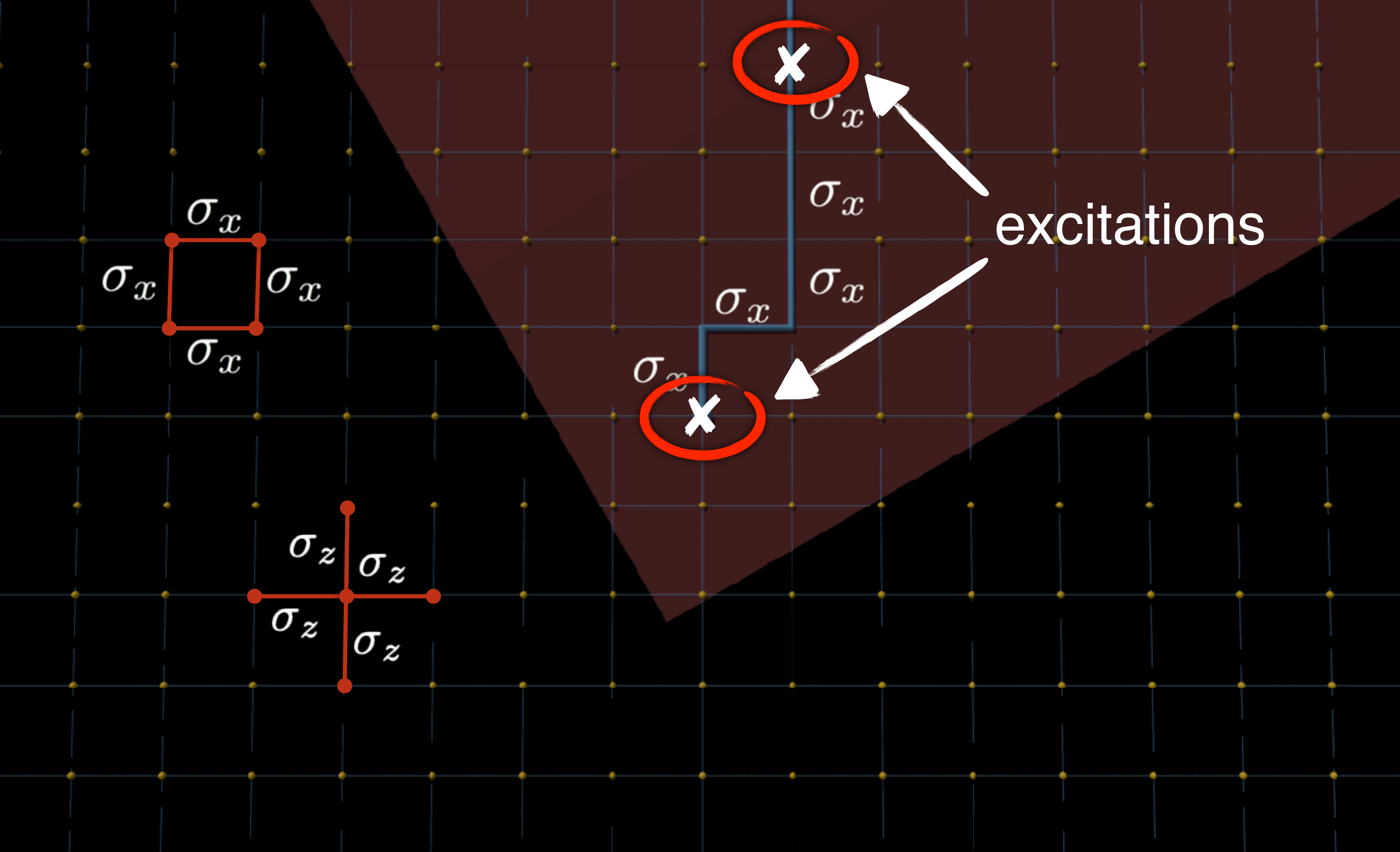
Image source: http://www.phy.anl.gov/theory/FritzFest/Fritz.html

Physical interpretation

This definition encodes the following properties:

1. **Charge conservation:** local operators cannot change the total charge
2. **Localisation** of the charge in cones
3. **Transportability** of the charges

Example: toric code



Example: toric code

$\omega_0 \circ \rho$ is a **single excitation** state

$$\rho(A) := \lim_{n \to \infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes observables in presence of **background charge**

Localised and transportable morphisms

The endomorphism ρ has the following properties:

- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$
- **transportable:** for Λ' there exists σ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these properties to find braiding, fusion, etc.

Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\operatorname{Rep} D(G)$.

Rev. Math. Phys. **23** (2011)
J. Math. Phys. **54** (2013)
Rev. Math. Phys. **27** (2015)

Stability

Classification of phases

- Does the gap stay open under small perturbations?

  Bravyi & Hastings, *J. Math. Phys.* **51** (2010)
  Michalakis & Zwolak, *Commun. Math. Phys.* **322** (2013)
  Nachtergaele, Sims & Young, arXiv:2102.07209
  and many others…

- How are the states related?

  Hastings, *Phys. Rev. B* **69** (2004)
  Hastings & Wen, *Phys. Rev. B* **72** (2005)
  Bachmann, Michalakis, Nachtergaele & Sims, *Commun. Math. Phys.* **309** (2012)
  Nachtergaele, Sims & Young, *J. Math. Phys.* **60** (2019)
  Moon & Ogata, *J. Funct. Anal.* **278** (2020)

Does this apply to the sector theory?

Theorem (Cha, PN, Nachtergaele)

Let *G* be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each *s* in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\mathrm{Rep}\, D(G)$.

Commun. Math. Phys. **373** (2020)

Long-range entanglement

Folklore

Topological order (and in particular anyonic excitations) are due to long-range entanglement

Long range entanglement

- Bipartite system $\mathfrak{A}_\Lambda \otimes \mathfrak{A}_{\Lambda^c}$
- Product states $\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}$ have only classical correlations
- LRE: $\omega \circ \alpha$ is not quasi-equivalent to a product state for any quasi-local automorphism
- In 1D, gapped ground states are not LRE, in 2D this can be different!

A new superselection criterion

We can relax the superselection criterion:

$$\pi \,|\, \mathfrak{A}_{\Lambda^c} \sim_{qe} \pi_\omega \,|\, \mathfrak{A}_{\Lambda^c}$$

That is, *quasi* instead of *unitary* equivalence

Remark: can be constructed naturally in non-abelian theories!

Szlachányi &Vecsernyés, CMP **156**, 1993

Theorem

Let ω be a pure state such that its GNS representation is quasi-equivalent to $\pi_\Lambda \otimes \pi_{\Lambda^c}$ for some cone Λ. Then the corresponding superselection structure is trivial.

PN & Y. Ogata, arXiv:2102.07707

The trivial phase

This shows that Kitaev's toric code cannot satisfy the split property

Can it still be in the same phase?

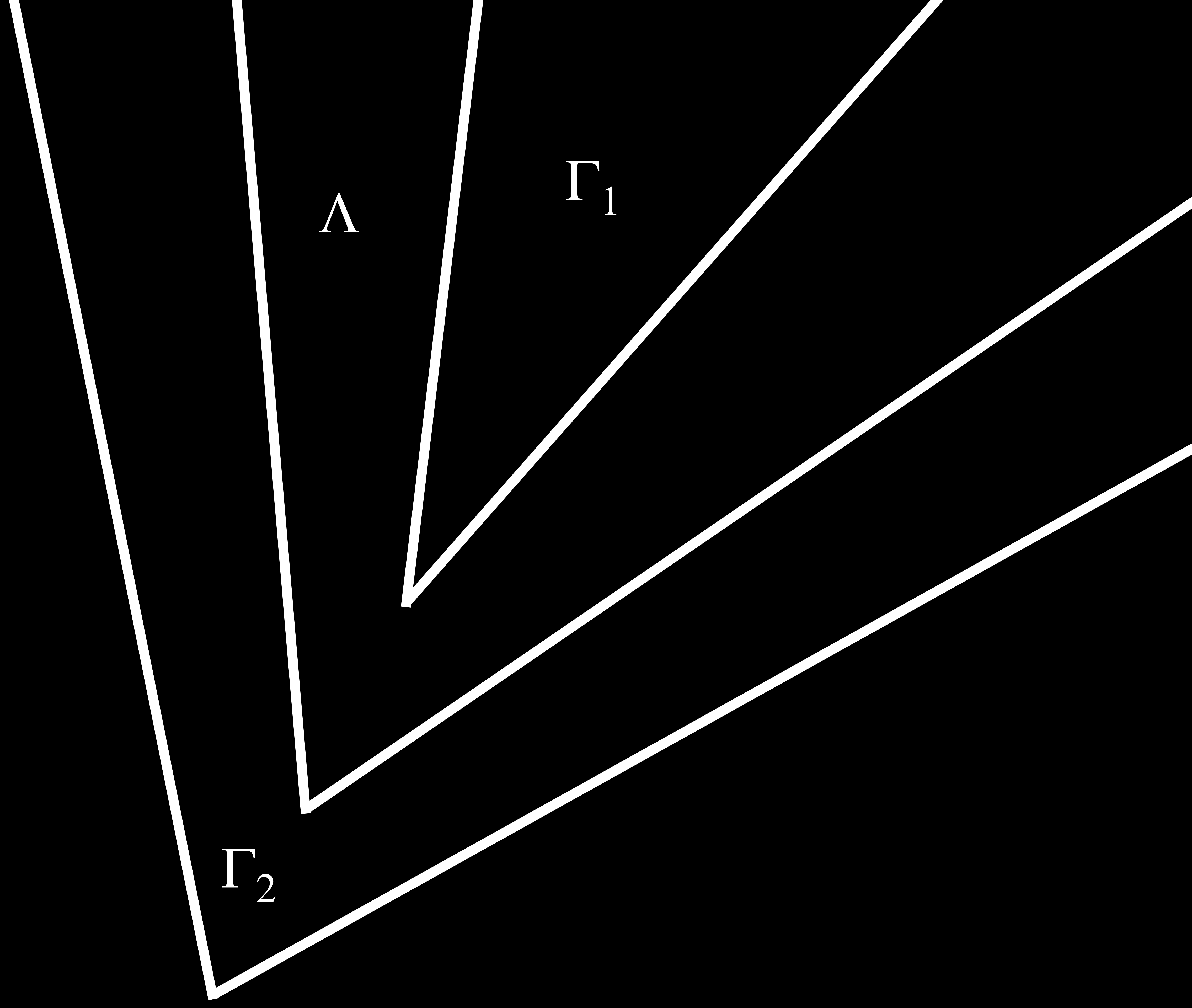
Definition

Consider an inclusion $\Gamma_1 \subset \Lambda \subset \Gamma_2$ of cones. Then $\alpha \in \mathrm{Aut}(\mathfrak{A})$ is called *quasi-factorisable* if:

$$\alpha = \mathrm{Ad}(u) \circ \Xi \circ (\alpha_\Lambda \otimes \alpha_{\Lambda^c})$$

for some unitary *u* and "local" automorphisms (see picture).

PN & Y. Ogata, arXiv:2102.07707



Λ
Γ_1
Γ_2

Theorem

Let π_0 be a representation and α quasi-factorisable for every cone. Then if π satisfies the selection criterion for π_0, then so does $\pi \circ \alpha$ for $\pi_0 \circ \alpha$.

Corollary

States in the trivial phase have trivial superselection structure.

PN & Y. Ogata, arXiv:2102.07707

Split property

Split property

The rigorous definition of many SPT invariants (in 1D) depends on the *split property*:

$$\pi_\omega(\mathfrak{A}_L)'' \subset \mathcal{N} \subset \pi_\omega(\mathfrak{A}_R)'$$

Equivalently, for ω pure:

$$\omega \sim_{qe} \omega_L \otimes \omega_R$$

Since $\mathcal{N}$ is a Type I factor: $\mathcal{H}_\omega \simeq \mathcal{H}_{\omega_L} \otimes \mathcal{H}_{\omega_R}$

Split property implies triviality of sector theory!

Split property

Theorem (Matsui, JMP 51, 2010)

A pure gapped ground state of a 1D spin chain satisfies the split property.

This is no longer true in 2D!

Theorem (PN, *Lett. Math. Phys.* 101, 2012)

The translational invariant ground state of the toric code satisfies the *approximate* split property, but not the split property.

The approximate split property

Approximate split property

Kitaev's abelian quantum double models satisfy a weaker form of the split property:

$$\pi(\mathfrak{A}_{\Lambda_1})'' \subset \mathcal{N} \subset \pi(\mathfrak{A}_{\Lambda_2})''$$

for suitable inclusions of cones $\Lambda_1 \subset \Lambda_2$

Interpretation: "entanglement is concentrated near the boundary of the cone"

Approximate split property is useful in stability analysis!

Theorem

The approximate split property is stable under quasi-local automorphisms.

PN & Y. Ogata, arXiv:2102.07707

Open problems

When do we have sectors?

Split property and TEE