

The split property and absence of superselection sectors in 2D

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arXiv:2102.07707 (with Yoshiko Ogata)

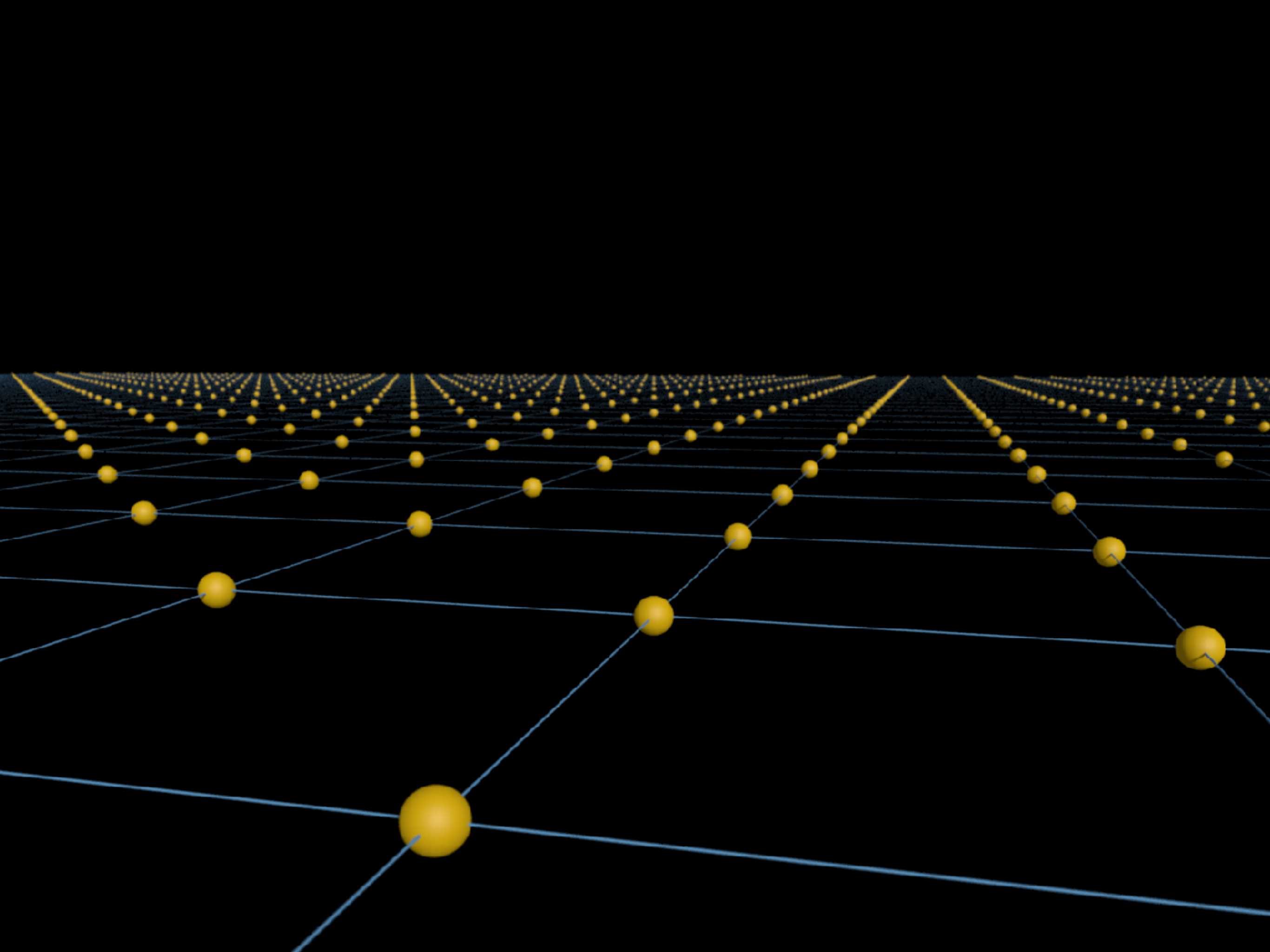
Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \text{spec}(H) \cap (0, \gamma) = \emptyset$$

Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

- > What are interesting phases?
- > Can we find invariants?

Quantum phases



Quantum spin systems

Consider 2D quantum spin systems, e.g. on \mathbb{Z}^2 :

> local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x \in \Lambda} M_d(\mathbb{C})$

> quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$

> local Hamiltonians H_Λ describing dynamics

> gives time evolution α_t & ground states

> if ω a ground state, Hamiltonian H_ω in GNS repn.

Quantum phases of ground states

Two ground states ω_0 and ω_1 are said to be *in the same phase* if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that ω_s is a ground state of $H(s)$.

(Chen, Gu, Wen, *Phys. Rev. B* **82**, 2010)

Alternative definition: ω_0 can be transformed into ω_1 with a *finite depth local quantum circuit*.



Classification of phases

> Does the gap stay open under small perturbations?

Bravyi & Hastings, *J. Math. Phys.* **51** (2010)

Michalakis & Zwolak, *Commun. Math. Phys.* **322** (2013)

Nachtergaele, Sims & Young, arXiv:2102.07209

and many others...

> How are the states related?

Hastings, *Phys. Rev. B* **69** (2004)

Hastings & Wen, *Phys. Rev. B* **72** (2005)

Bachmann, Michalakis, Nachtergaele & Sims, *Commun. Math. Phys.* **309** (2012)

Nachtergaele, Sims & Young, *J. Math. Phys.* **60** (2019)

Moon & Ogata, *J. Funct. Anal.* **278** (2020)

Can we find invariants?

Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

Invariants

Symmetry protected phases

1D case is best understood:

- > Time-reversal symmetry: \mathbb{Z}_2 index
- > On-site group symmetry, $H^2(G, U(1))$
- > Further refinements for e.g. fermionic systems
- > Recently: extensions to 2D

Kapustin, Sopenko, Yang, arXiv:2012.15491

Bourne & Ogata, *Forum of Mathematics, Sigma* **9** (2021)

Ogata, arXiv:2101.00426 (2021)

Sopenko, arXiv:2101.00801 (2021)

Split property

The rigorous definition of the invariants (in ID) depends on the *split property*:

$$\pi_\omega(\mathfrak{A}_L)'' \subset \mathcal{N} \subset \pi_\omega(\mathfrak{A}_R)'$$

Equivalently, for ω pure:

$$\omega \sim_{qe} \omega_L \otimes \omega_R$$

Since \mathcal{N} is a Type I factor: $\mathcal{H}_\omega \simeq \mathcal{H}_{\omega_L} \otimes \mathcal{H}_{\omega_R}$

Split property

Theorem (Matsui, JMP 51, 2010)

A pure gapped ground state of a 1D spin chain satisfies the split property.

This is no longer true in 2D!

Theorem (PN, *Lett. Math. Phys.* 101, 2012)

The translational invariant ground state of the toric code satisfies the *approximate* split property, but not the split property.

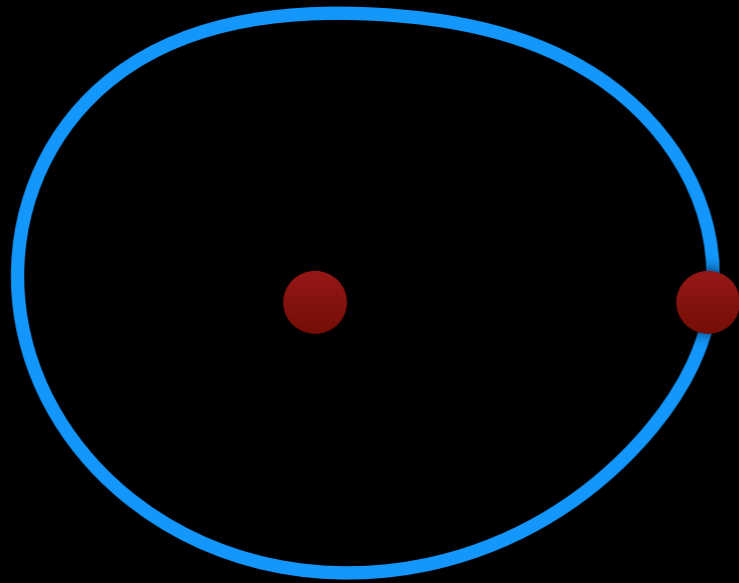
Topological order

Topological phases

Quantum phase outside of Landau theory

- > physics depends on topological properties
- > ground space degeneracy
- > long range entanglement
- > anyonic excitations
- > modular tensor category / TQFT

Anyons in 2D



not contractible!

In quantum mechanics (abelian case):

$$\psi \rightarrow e^{i\theta} \psi$$

Anyons and modular tensor categories

anyon types \Leftrightarrow irreducible objects

$$\text{fusion of charges } \Leftrightarrow \rho_i \otimes \rho_j = \sum_k N_{ij}^k \rho_k$$

conjugate charge \Leftrightarrow duals/conjugates

exchanging anyons \Leftrightarrow braiding

detect anyons through braiding \Leftrightarrow modularity

Mathematical connections

- > Subfactor theory
- > Higher category theory
- > Topological quantum field theory
- > Conformal field theory
- > And many more...

Problems:

How to get the MTC?

Is this an invariant?

LRE and trivial phases

Sector theory



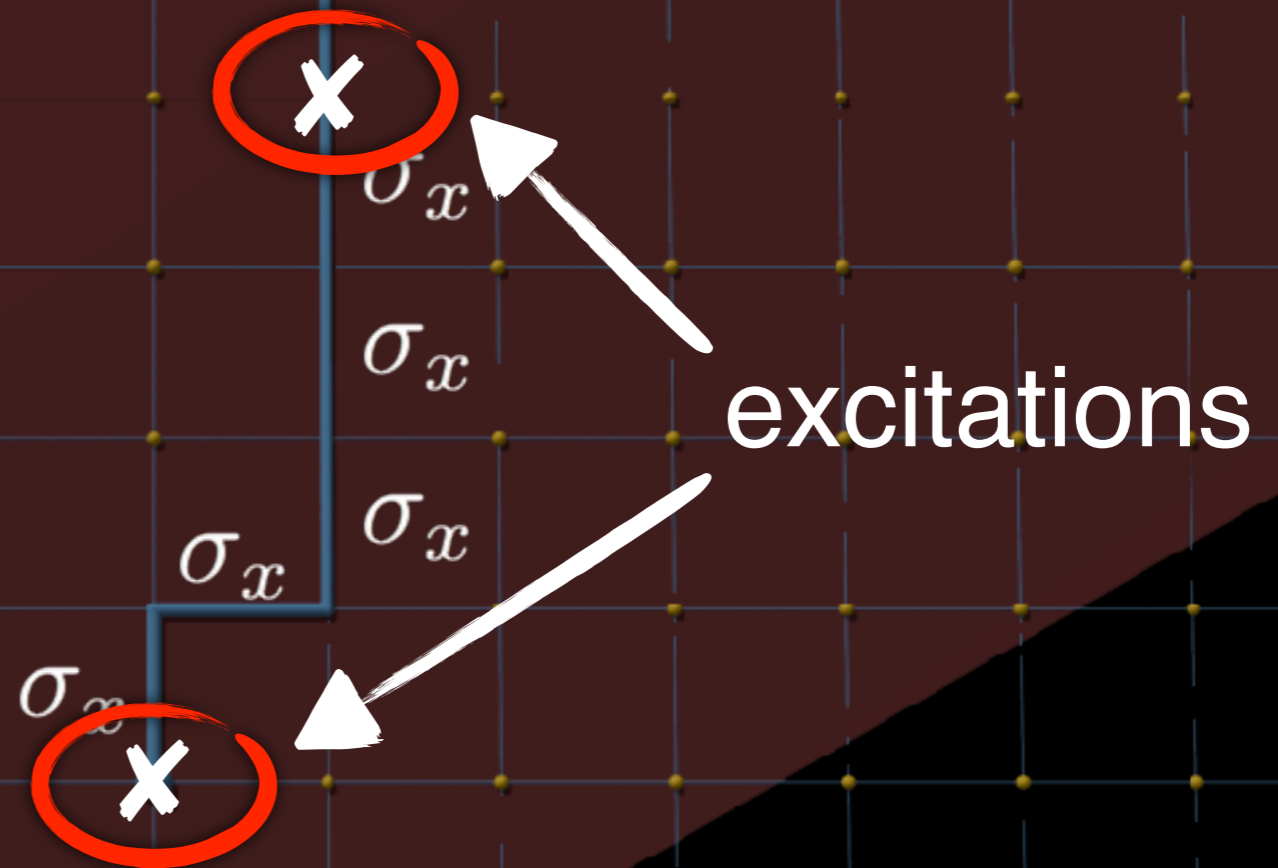
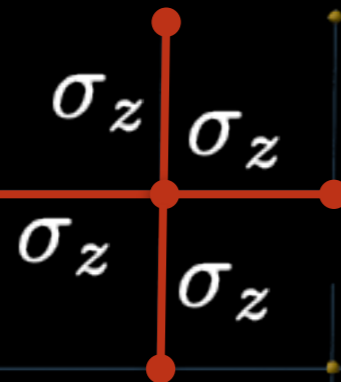
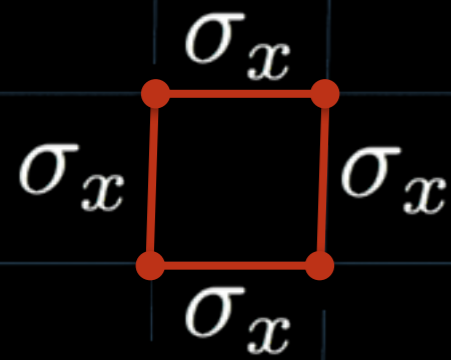
Definition

A **superselection sector** is an equivalence class of representations π such that

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

for all cones Λ .

Example: toric code



Example: toric code

$\omega_0 \circ \rho$ is a **single excitation state**

$$\rho(A) := \lim_{n \rightarrow \infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes
observables in
presence of
background charge

Localised and transportable morphisms

The endomorphism ρ has the following properties:

> **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$

> **transportable:** for Λ' there exists σ localised
and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these
properties

Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\text{Rep } D(G)$.

Rev. Math. Phys. **23** (2011)

J. Math. Phys. **54** (2013)

Rev. Math. Phys. **27** (2015)

Automorphic equivalence

Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi_\Lambda(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

Quasi-locality

The main feature about the family of automorphisms α_s is that they are **quasi-local**, i.e. satisfy a **Lieb-Robinson** type of bound:

$$\|[\alpha(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F} (e^{C_\Phi} - 1) |X| G_F(d(X, Y))$$

This implies good localisation properties for α !

Theorem

Let G be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each s in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\text{Rep } D(G)$.

Cha, PN, Nachtergaele, *Commun. Math. Phys.* **373** (2020)

Long range entanglement

Long range entanglement

- > Bipartite system $\mathfrak{A}_\Lambda \otimes \mathfrak{A}_{\Lambda^c}$
- > Product states $\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}$ have only classical correlations
- > LRE: $\omega \circ \alpha$ is not quasi-equivalent to a product state for any quasi-local automorphism
- > In 1D, gapped ground states are not LRE, in 2D this can be different!

Folklore

Topological order (and in particular anyonic excitations) are due to long range entanglement

A new superselection criterion

We can relax the superselection criterion:

$$\pi | \mathfrak{A}_{\Lambda^c} \sim_{qe} \pi_{\omega} | \mathfrak{A}_{\Lambda^c}$$

That is, *quasi* instead of *unitary* equivalence

Remark: can be constructed naturally in non-abelian theories!

Theorem

Let ω be a pure state such that its GNS representation is quasi-equivalent to $\pi_{\Lambda} \otimes \pi_{\Lambda^c}$ for some cone Λ . Then the corresponding superselection structure is trivial.

The trivial phase

This shows that Kitaev's toric code cannot satisfy the split property

Can it still be in the same phase?

Definition

Consider an inclusion $\Gamma_1 \subset \Lambda \subset \Gamma_2$ of cones. Then $\alpha \in \text{Aut}(\mathfrak{A})$ is called *quasi-factorisable* if:

$$\alpha = \text{Ad}(u) \circ \Xi \circ (\alpha_\Lambda \otimes \alpha_{\Lambda^c})$$

for some unitary u and “local” automorphisms (see picture).

Λ

Γ_1

Γ_2

Theorem

Let π_0 be a representation and α quasi-factorisable for every cone. Then if π satisfies the selection criterion for π_0 , then so does $\pi \circ \alpha$ for $\pi_0 \circ \alpha$.

Corollary

States in the trivial phase have trivial superselection structure.

The technical ingredients

Proof sketch

- > Split property/Type I factor give factorisation of Hilbert space wrt. cone
- > Superselection criterion implies similar factorisation for representation π (up to amplification)
- > Quasi-factorisability preserves locality properties in suitable sense
- > Then show that suitable α exist!

Theorem 3.1. *Let (Γ, d) be a countable ν -regular metric space with constant κ as in (2.1). Let F be an F -function on (Γ, d) such that the function G_F defined by (2.19) satisfies (2.34) for some $\alpha \in (0, 1)$. Suppose that there is an F -function \tilde{F} satisfying (2.35) for this F . Let \mathcal{A}_Γ be a quantum spin system given by (2.3) and (2.4).*

Let $\Phi \in \mathcal{B}_F([0, 1])$ be a path of interactions satisfying $\Phi_1 \in \mathcal{B}_F([0, 1])$. (Recall from definition (2.22) that this means that $X \mapsto |X|\Phi(X; t)$ is in $\mathcal{B}_F([0, 1])$). Let

$$\Gamma'_1 \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma'_2 \subset \Gamma. \quad (3.1)$$

For $m \in \mathbb{N} \cup \{0\}$, $x, y \in \Gamma$, set

$$f(m, x, y) := \sum_{X \ni x, y, d((\Gamma'_2 \setminus \Gamma'_1)^c, X) \leq m} |X| \sup_{t \in [0, 1]} \|\Phi(X, t)\|. \quad (3.2)$$

We assume that

$$\left(\sum_{x \in \Gamma_1} \sum_{y \in \Gamma_2^c} + \sum_{x \in \Gamma_2 \setminus \Gamma_1} \sum_{y \in (\Gamma_2 \setminus \Gamma_1)^c} \right) \sum_{m=0}^{\infty} G_F(m) f(m, x, y) < \infty \quad (3.3)$$

Define $\Phi^{(0)} \in \mathcal{B}_F([0, 1])$ by

$$\Phi^{(0)}(X; t) := \begin{cases} \Phi(X; t), & \text{if } X \subset \Gamma_1 \text{ or } X \subset \Gamma_2 \setminus \Gamma_1 \text{ or } X \subset \Gamma_2^c, \\ 0, & \text{otherwise} \end{cases}, \quad (3.4)$$

for each $X \in \mathcal{P}_0(\Gamma)$, $t \in [0, 1]$. Then there is an automorphism $\beta_{\Gamma'_2 \setminus \Gamma'_1}$ on $\mathcal{A}_{\Gamma'_2 \setminus \Gamma'_1}$ and a unitary $u \in \mathcal{A}_\Gamma$ such that

$$\tau_{1,0}^\Phi = \text{Ad}(u) \circ \tau_{1,0}^{\Phi^{(0)}} \circ \left(\tilde{\beta}_{\Gamma'_2 \setminus \Gamma'_1} \right). \quad (3.5)$$

Conditions can be checked
in relevant examples!

The approximate split property

Approximate split property

Kitaev's abelian quantum double models satisfy a weaker form of the split property:

$$\pi(\mathfrak{A}_{\Lambda_1})'' \subset \mathcal{N} \subset \pi(\mathfrak{A}_{\Lambda_2})''$$

for suitable inclusions of cones $\Lambda_1 \subset \Lambda_2$

Interpretation: “entanglement is concentrated near the boundary of the cone”

Approximate split property is useful in stability analysis!

Theorem

The approximate split property is stable under quasi-local automorphisms.

Open problems

Full sector theory

When do we have sectors?

Split property and TEE