Topological phases of matter in an operator-algebraic approach

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Operator Algebras: Subfactors, K-theory, CFT
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Topological phases of matter

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

- What are interesting phases?
- Can we find invariants?

Topological insulators

Altland-Zirnbauer classes:
builds on work by **Cartan**

homotopy groups

Cartan						*d*							
	0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

Kitaev, AIP Conference Proceedings **1134**, 2009; Ryu *et al, New J. Phys.* **12**, 2010

Generalisations

- Disorder, interacting fermions

  Non-commutative geometry, *K*-theory for operator algebras, index theory

- Symmetry Protected Topological (SPT) phases

  Classified by group cohomology $H^{\nu+1}(G, \mathbb{T})$ (in dim. $\nu = 1{,}2$). Recent rigorous results by Ogata using operator-algebraic techniques.

Topological order

Quantum phase outside of Landau theory

- No good definition known
- ground space degeneracy
- long range entanglement
- anyonic excitations

Wen, *Int. J. Mod. Phys. B.* **4**, 1990

Folklore

The anyonic quasi-particle excitations of a topologically ordered state are described by a modular tensor category.

Kitaev, *Ann. Physics* **321**,2006

Examples

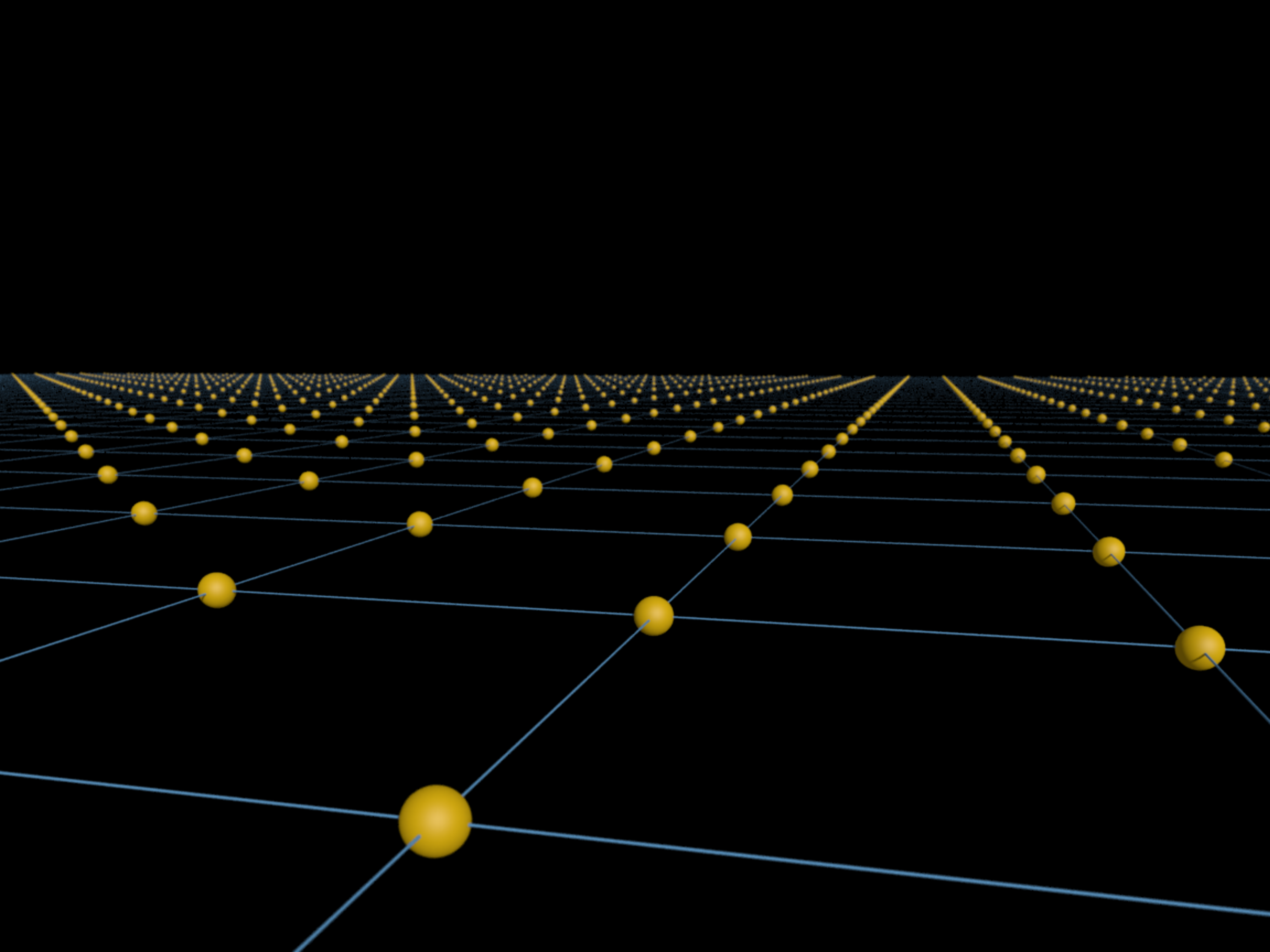
- Kitaev quantum double models, honeycomb
- Tensor network states (PEPS)
- Levin-Wen models
- TQFT approach
- Category theory approach
- Entanglement properties

Wishlist

- From first principles / minimal assumptions
- Result in braided tensor category
- Stable under deformations (i.e., gives rise to proper invariants)
- Mathematically rigorous
- Include relevant examples

An operator algebraic approach

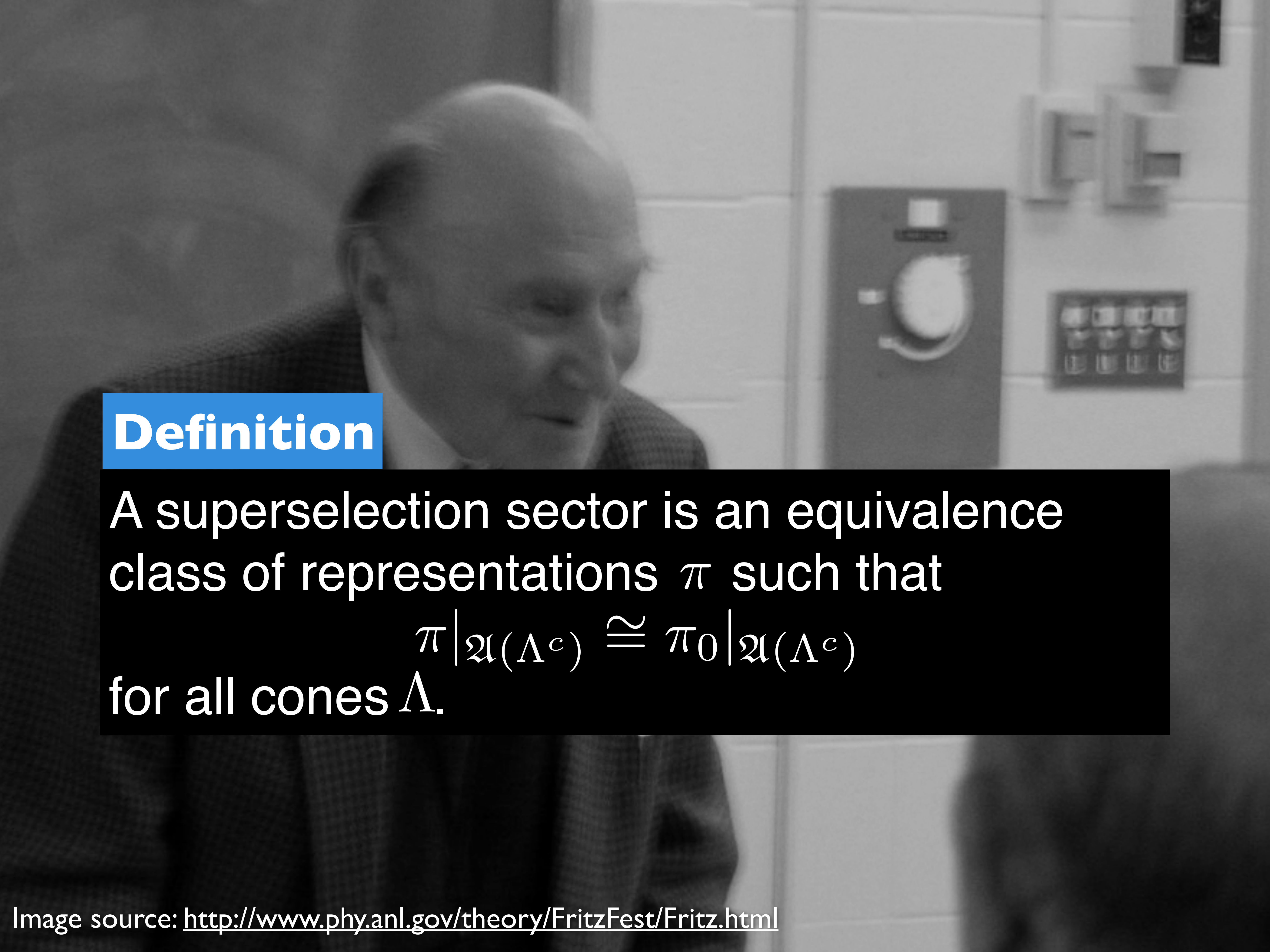
(Motivated by DHR theory)



Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x \in \Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution α_t & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.
- ω is called a *gapped GS* if H_ω is gapped



Definition

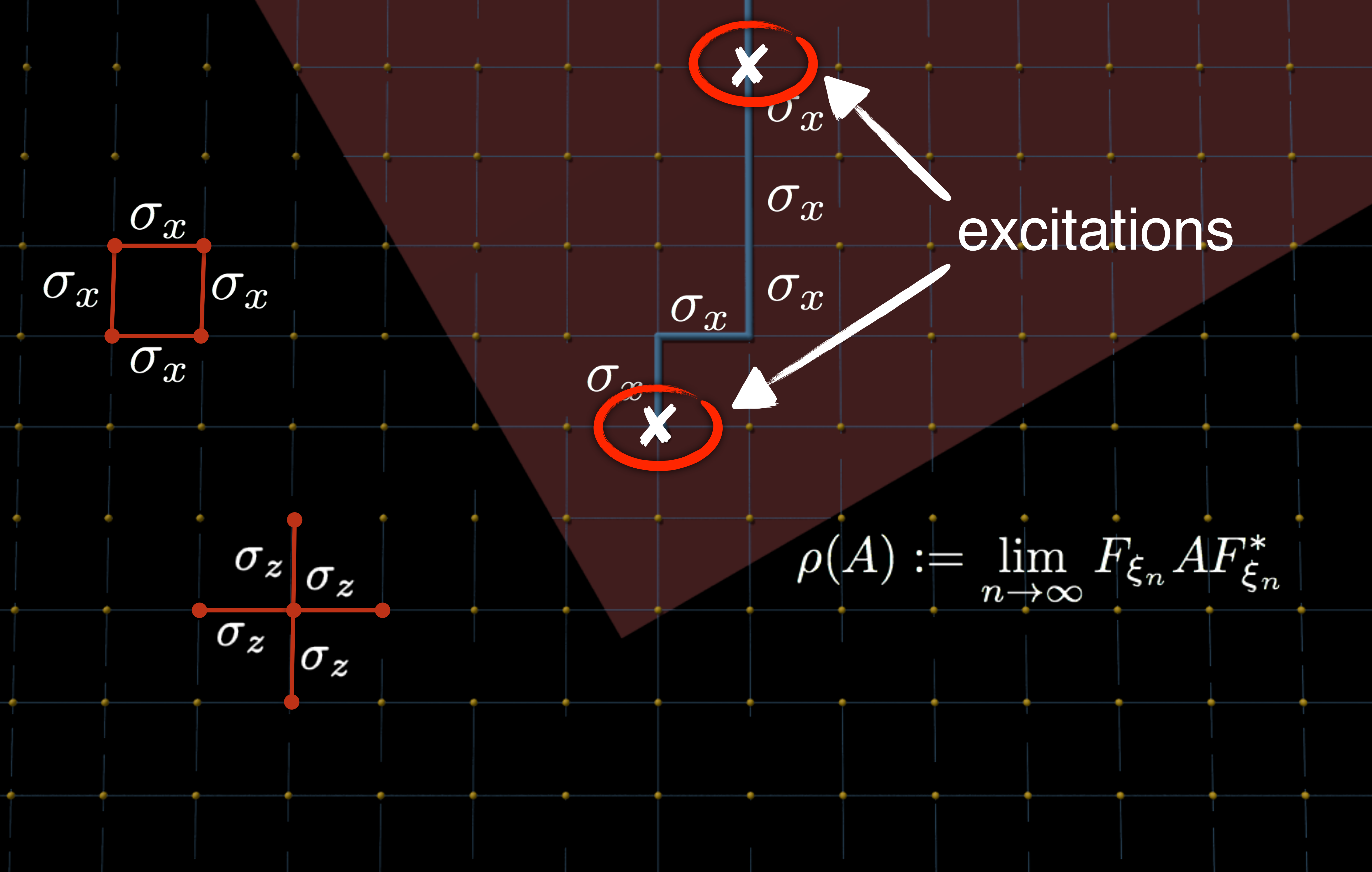
A superselection sector is an equivalence class of representations π such that

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

for all cones Λ.

Image source: http://www.phy.anl.gov/theory/FritzFest/Fritz.html

Example: toric code



$$\rho(A) := \lim_{n\to\infty} F_{\xi_n} A F^*_{\xi_n}$$

Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\mathrm{Rep}\, D(G)$.

Rev. Math. Phys. **23** (2011)
J. Math. Phys. **54** (2013)
Rev. Math. Phys. **27** (2015)

Key step: replace irreps

Use endomorphisms ρ with the following properties:

- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$
- **transportable:** for Λ' there exists σ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these properties

Comparison with CFT

CFT	Quantum spin
Local algebras Type III	Local algebras finite dimensional
Haag duality	Approximate Haag duality
Reeh-Schlieder property & Möbius covariance	Not separating or cyclic for local algebras

Equivalence of states

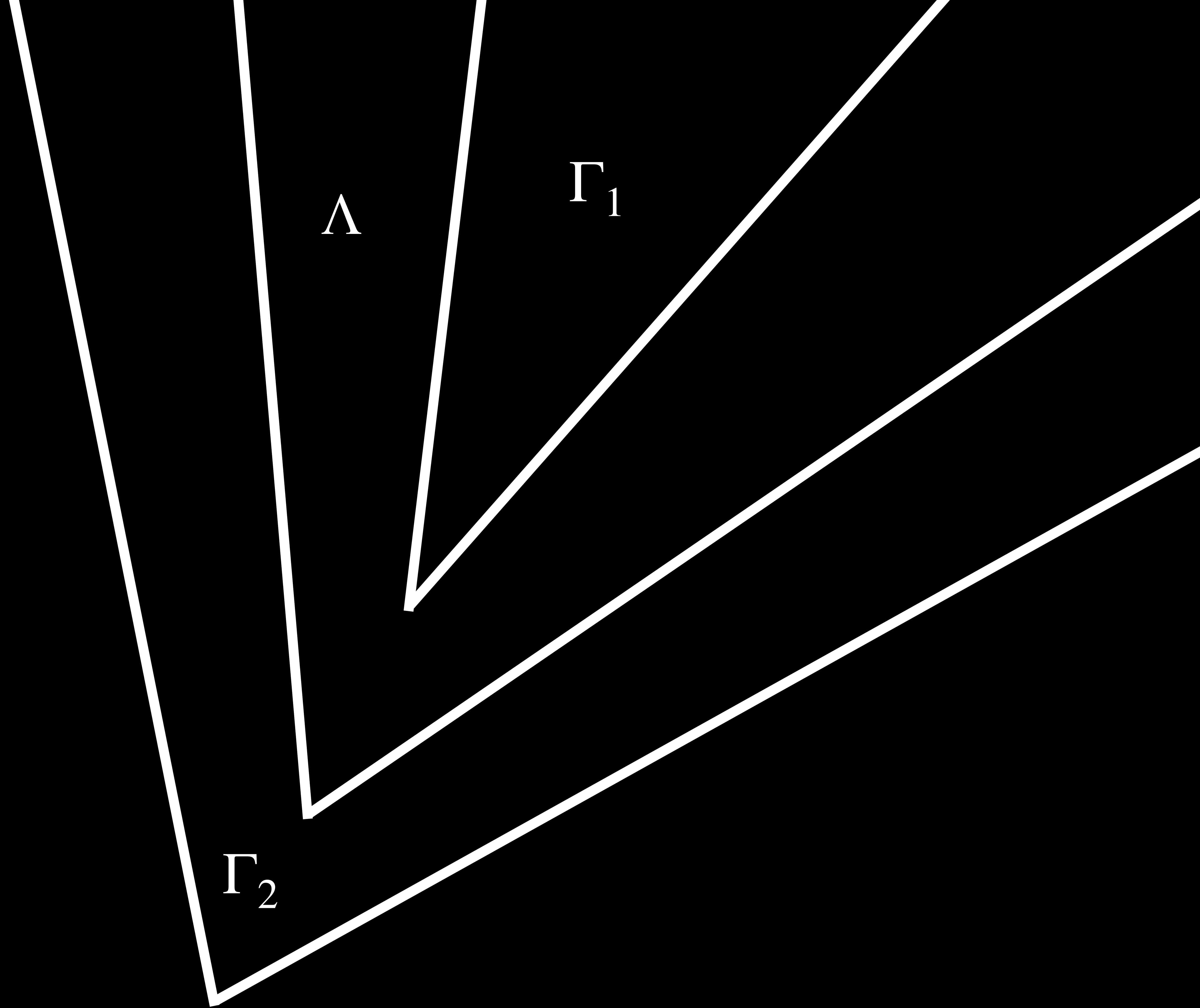
Definition

Consider an inclusion $\Gamma_1 \subset \Lambda \subset \Gamma_2$ of cones. Then $\alpha \in \mathrm{Aut}(\mathfrak{A})$ is called *quasi-factorisable* if

$$\alpha = \mathrm{Ad}(u) \circ \Xi \circ (\alpha_\Lambda \otimes \alpha_{\Lambda^c})$$

for some unitary $u \in \mathfrak{A}$ and "local" automorphisms (see picture).

PN & Y. Ogata, *Commun. Math. Phys.* **392** (2022)


Λ
Γ1
Γ2

Equivalence of phases

Two states are in the same phase if there is a quasi-factorisable (for any cone) automorphism α such that $\omega_0 = \omega_1 \circ \alpha$.
Can be constructed using LR bounds:

$$\|[\alpha(A), B]\| \leq \frac{2\|A\|\,\|B\|}{C_F}\left(e^{C_\Phi} - 1\right)|X|\,G_F(d(X,Y))$$

Such automorphisms can be obtained naturally from suitable gapped paths of local Hamiltonians!

Approximate Haag duality

Some parts of the sector theory use Haag duality:

$$\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$$

Not obvious this holds for $\pi_0 \circ \alpha$!

Better notion: *approximate* Haag duality:

$$\pi_0(\mathfrak{A}(\Lambda^c))' \subset U_{\Lambda,\epsilon}\pi_0(\mathfrak{A}(\Lambda_\epsilon))U_{\Lambda,\epsilon}^*$$

Ogata, *J. Math. Phys.* **63** (2022)

Theorem

Let G be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each s in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\mathrm{Rep}\, D(G)$.

Cha, PN, Nachtergaele, *Commun. Math. Phys.* **373** (2020)
Ogata, *J. Math. Phys.* **63** (2022)

Cone algebras

Relevant algebras: cone algebras $\pi_0(\mathfrak{A}(\Lambda))''$.
These are factors!

- Type I factor: sector theory is trivial (need LRE!)

  PN & Ogata, *Commun. Math. Phys.* **392** (2022)

- Otherwise, Type II_∞ or Type III

  PN, *Lett. Math. Phys.* **101** (2012)
  Ogata, *J. Math. Phys.* **63** (2022)

But, if $p \in \mathrm{Hom}(\rho, \rho)$ projection, then $p \in \pi_0(\mathfrak{A}(\Lambda))''$ and there is $v \in \pi_0(\mathfrak{A}(\Lambda))''$ with $v^*v = I, vv^* = p$.

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Conclusions

The AQFT / local conformal net-inspired approach to superselection sectors in topologically ordered phases provides an elegant framework to study phases, based on first principles and a microscopic description of the system.

Questions:

When is it modular?

Interesting models?

Various constructions