Ground states of Kitaev's quantum double model in the thermodynamic limit

Pieter Naaijkens

Cardiff University

Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

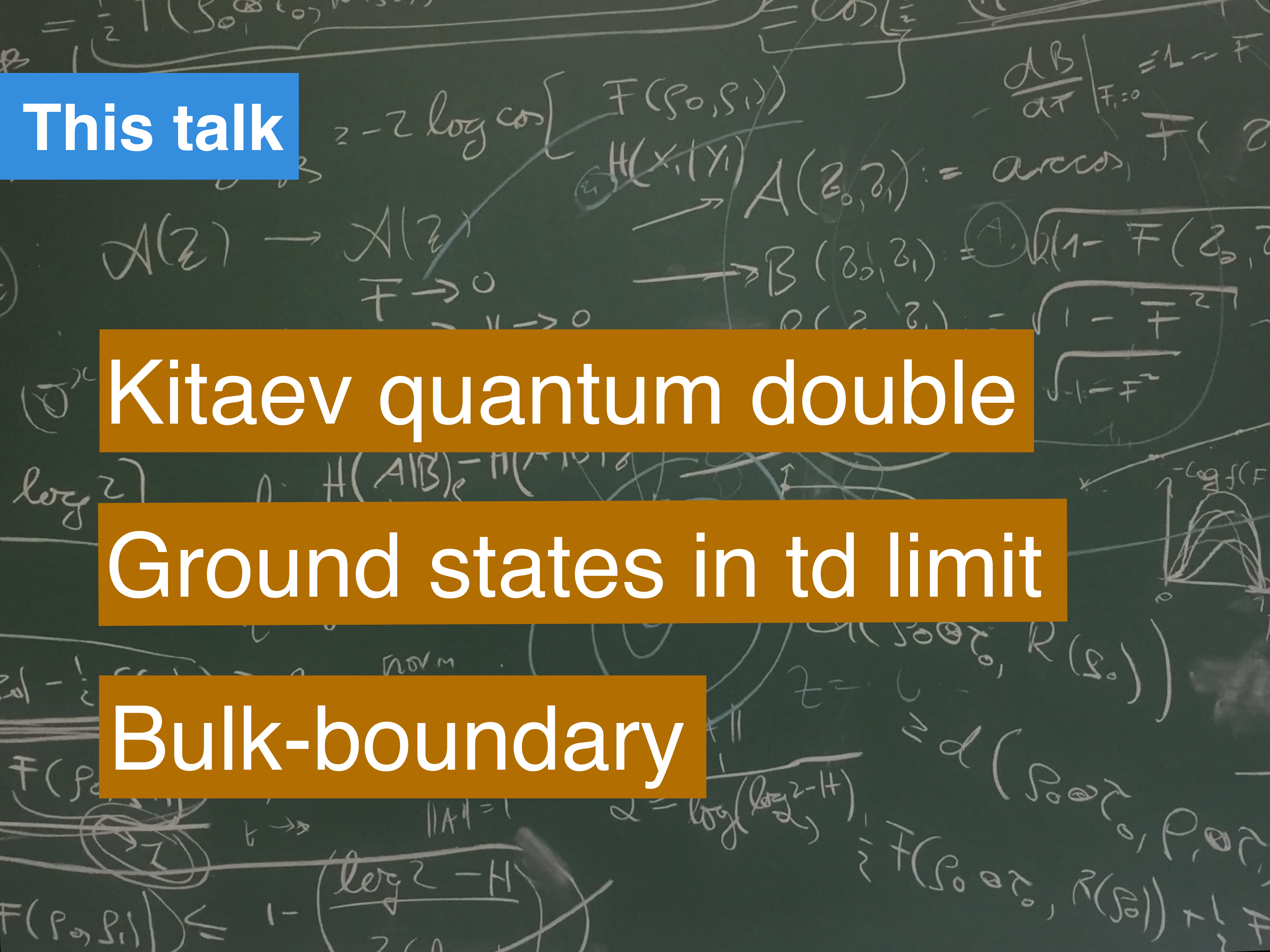
Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

Topological order

Quantum phase outside of Landau theory

- No good definition known
- Ground space degeneracy depending on topology
- Long range entanglement
- Anyonic excitations
- Bulk-boundary

Wen, *Int. J. Mod. Phys. B.* **4**,1990



This talk

Kitaev quantum double

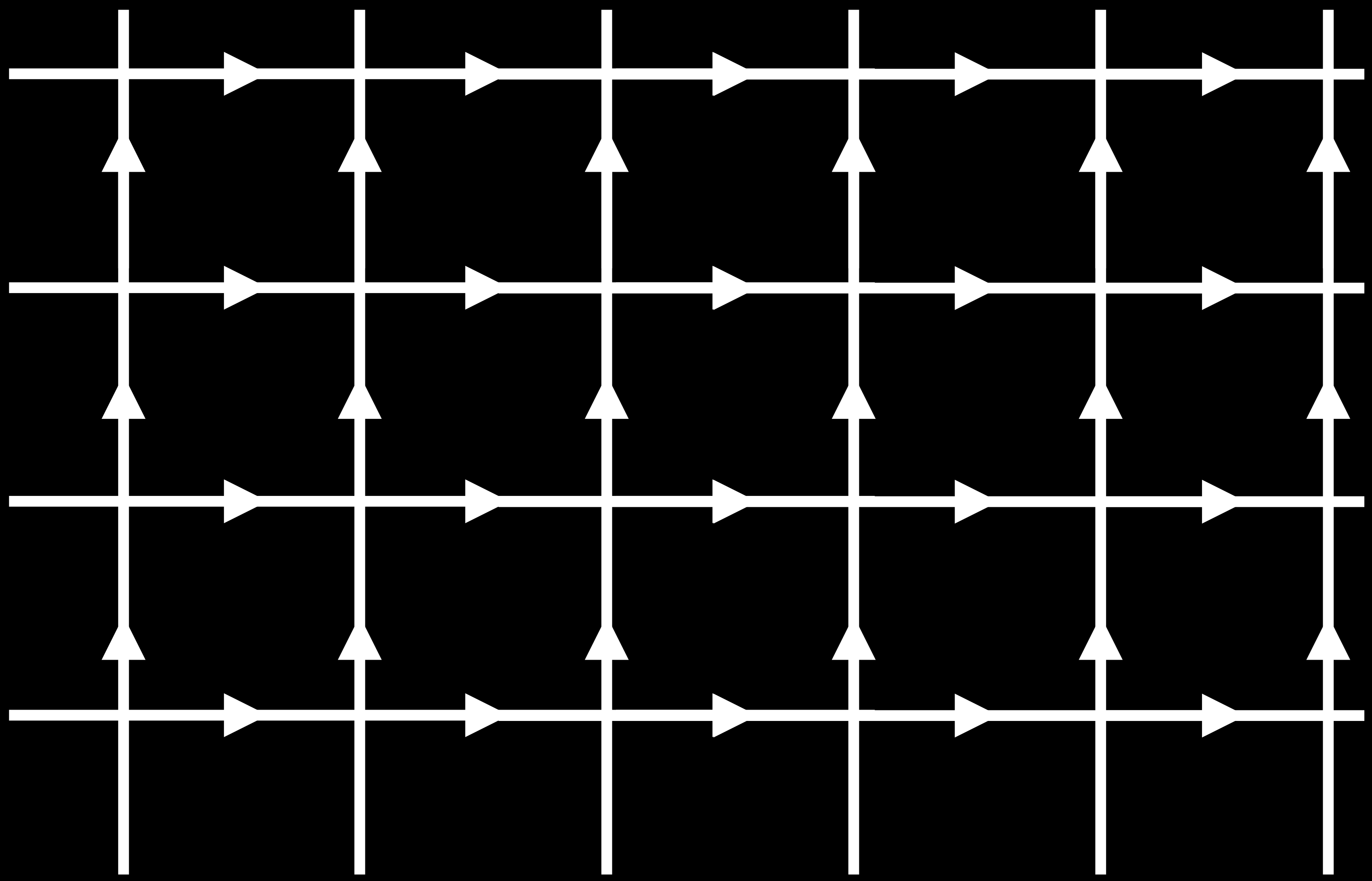
Ground states in td limit

Bulk-boundary

Kitaev’s quantum double model

Kitaev quantum double

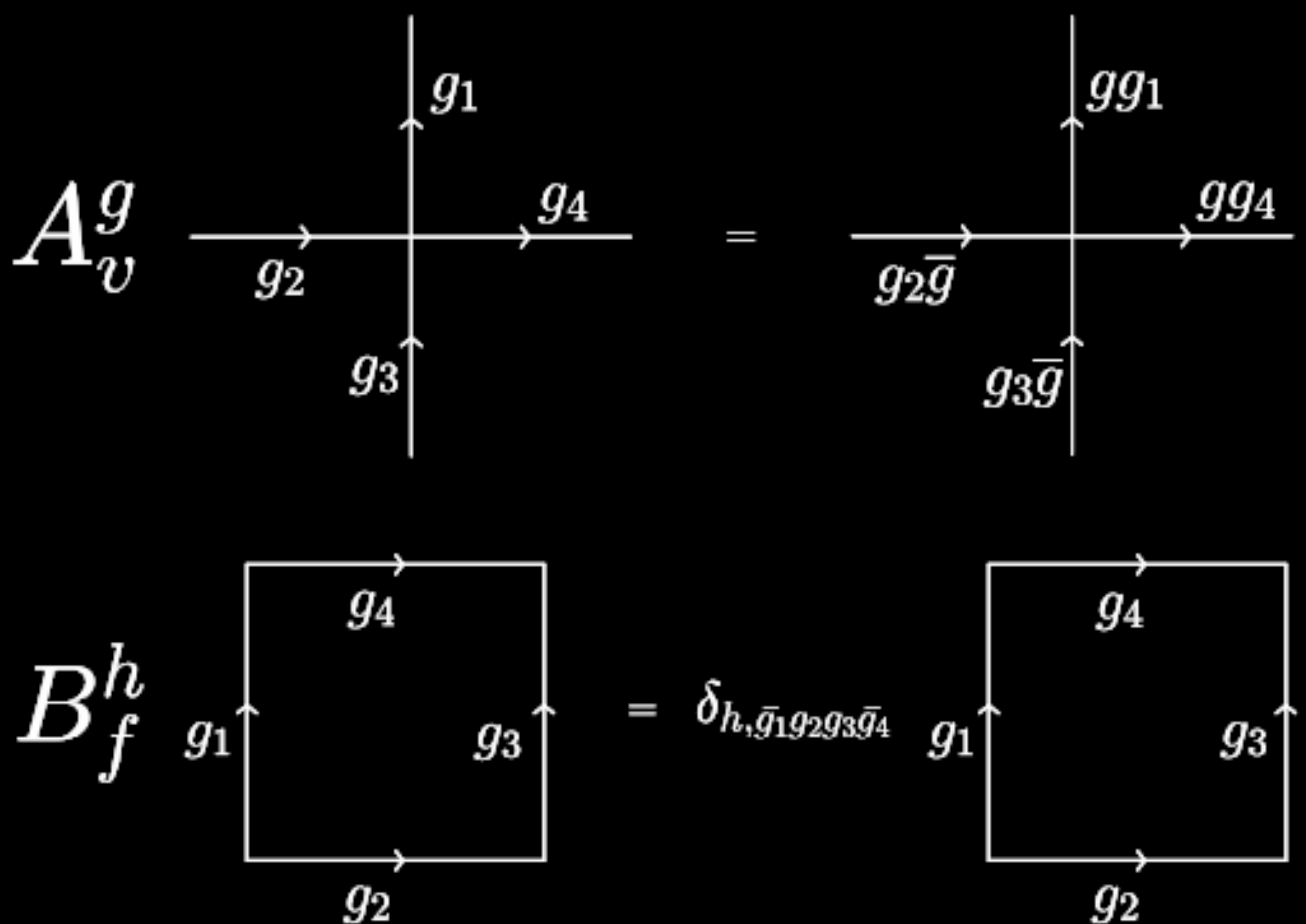
In the following, G is a finite group



$$\mathscr{H}_e = \ell^2(G)$$

Kitaev, *Ann. Physics* **303**,2003

Star and plaquette operators

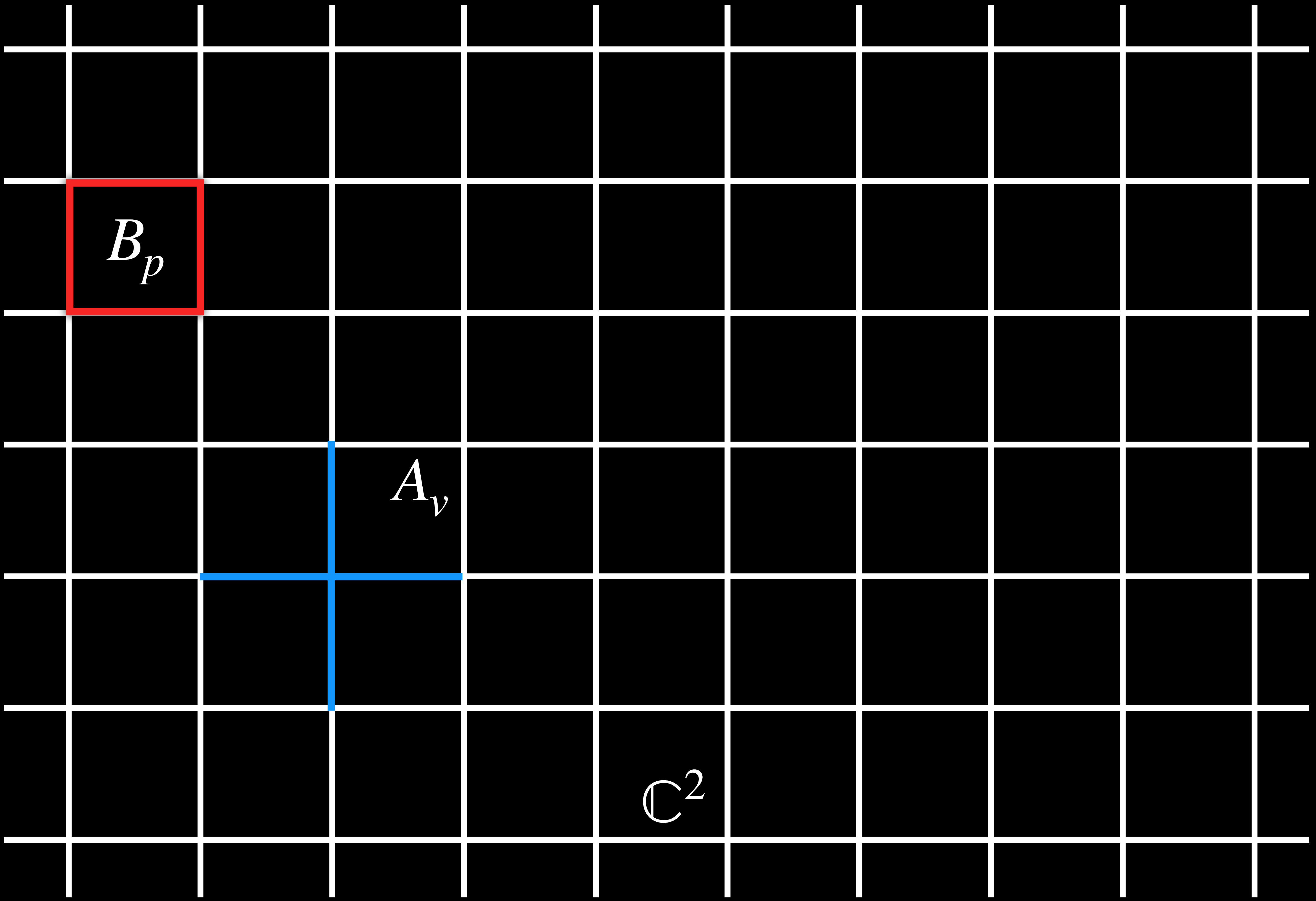


$$A_v := \frac{1}{|G|}\sum_{g\in G} A_v^g \qquad B_p := B_p^e$$

$$[A_v, B_p] = [A_v, A_{v'}] = [B_p, B_{p'}] = 0$$

Kitaev, *Ann. Physics* **303**,2003

Ground states



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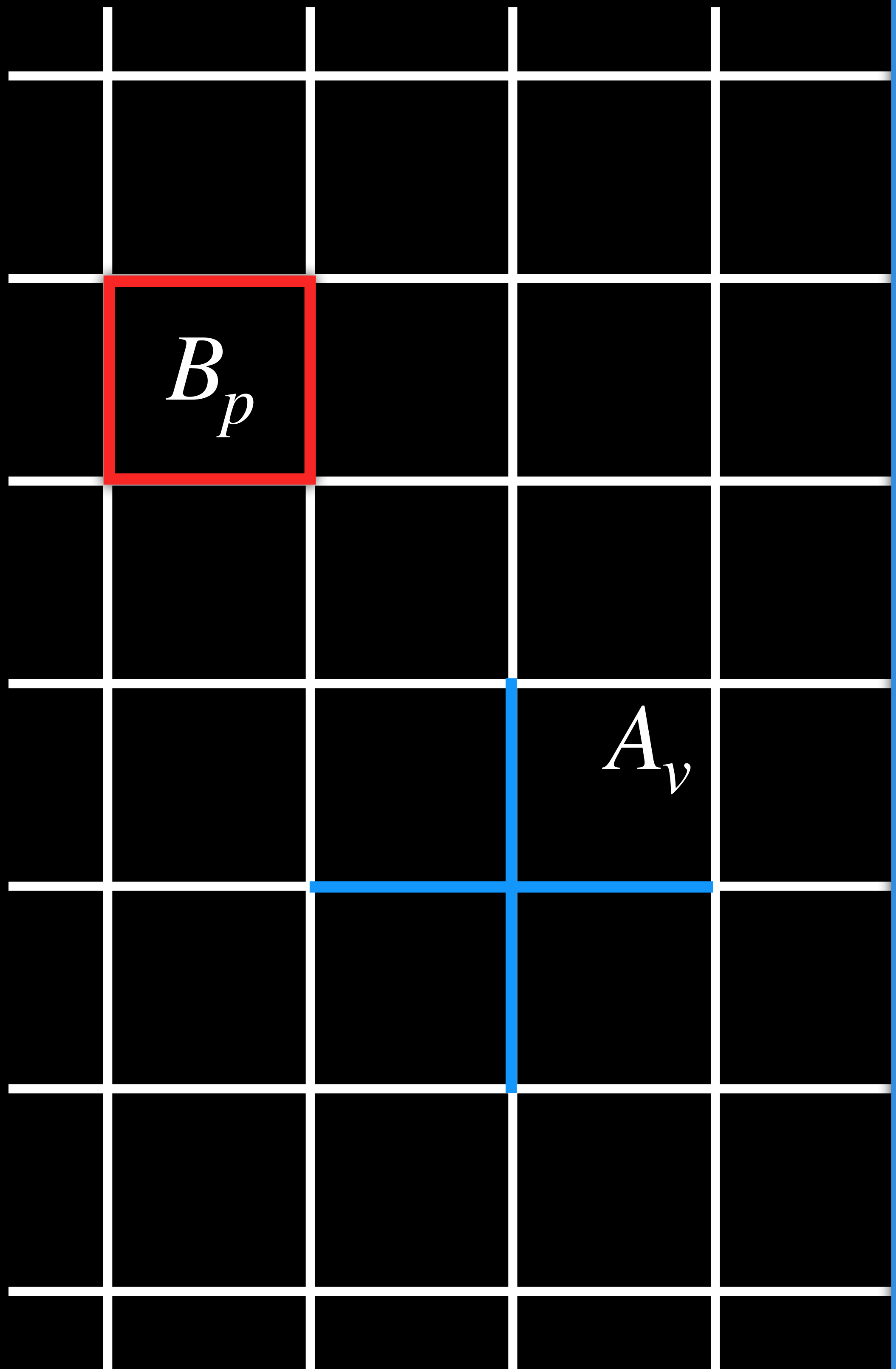


Hamiltonian:

$$H = \sum_v (I - A_v) + \sum_p (I - B_p)$$

Ground state:

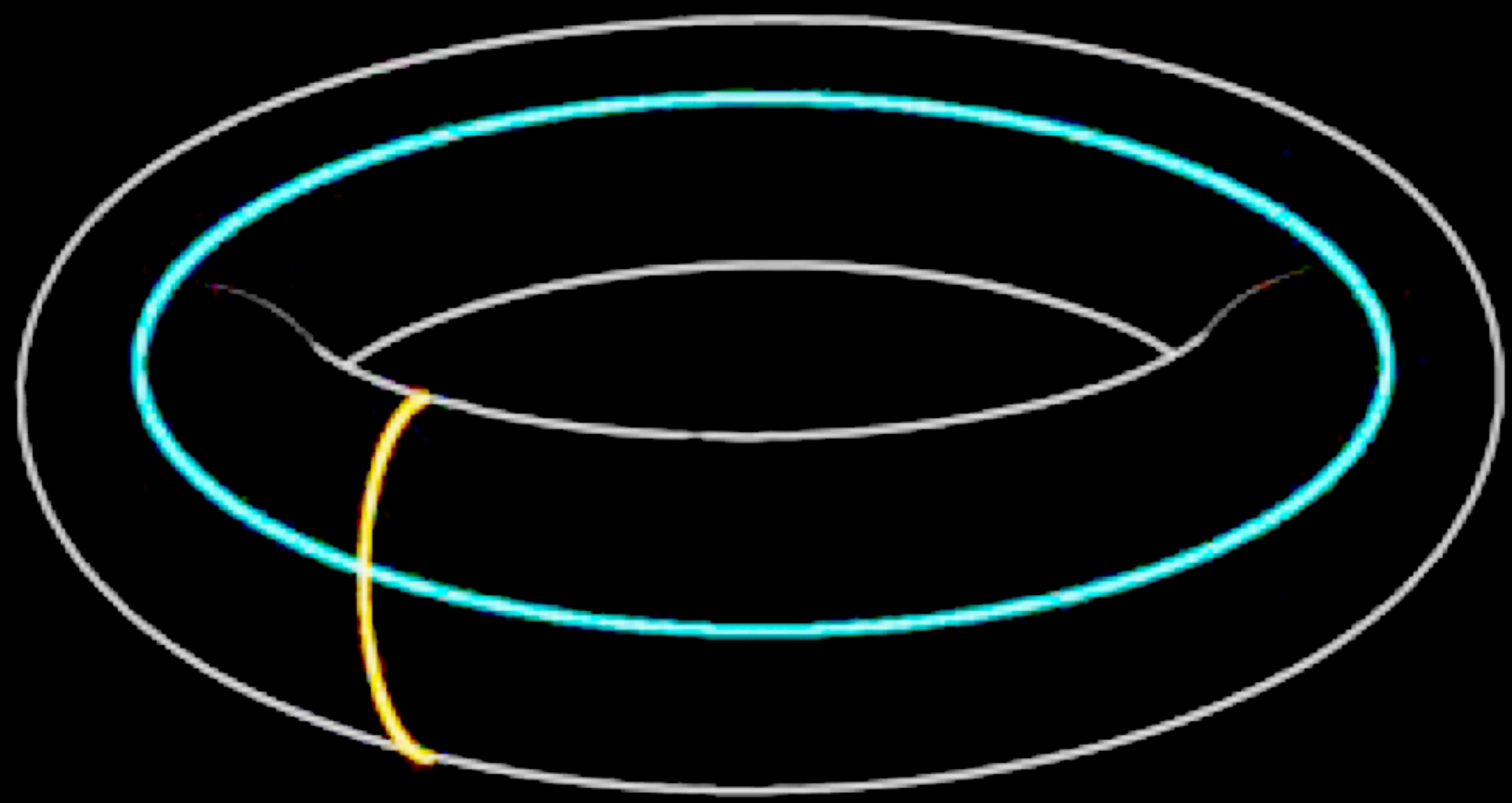
$$A_v \Omega = B_p \Omega = \Omega$$

Remark: on compact surface, degeneracy depends on genus (for toric code, 4^g)

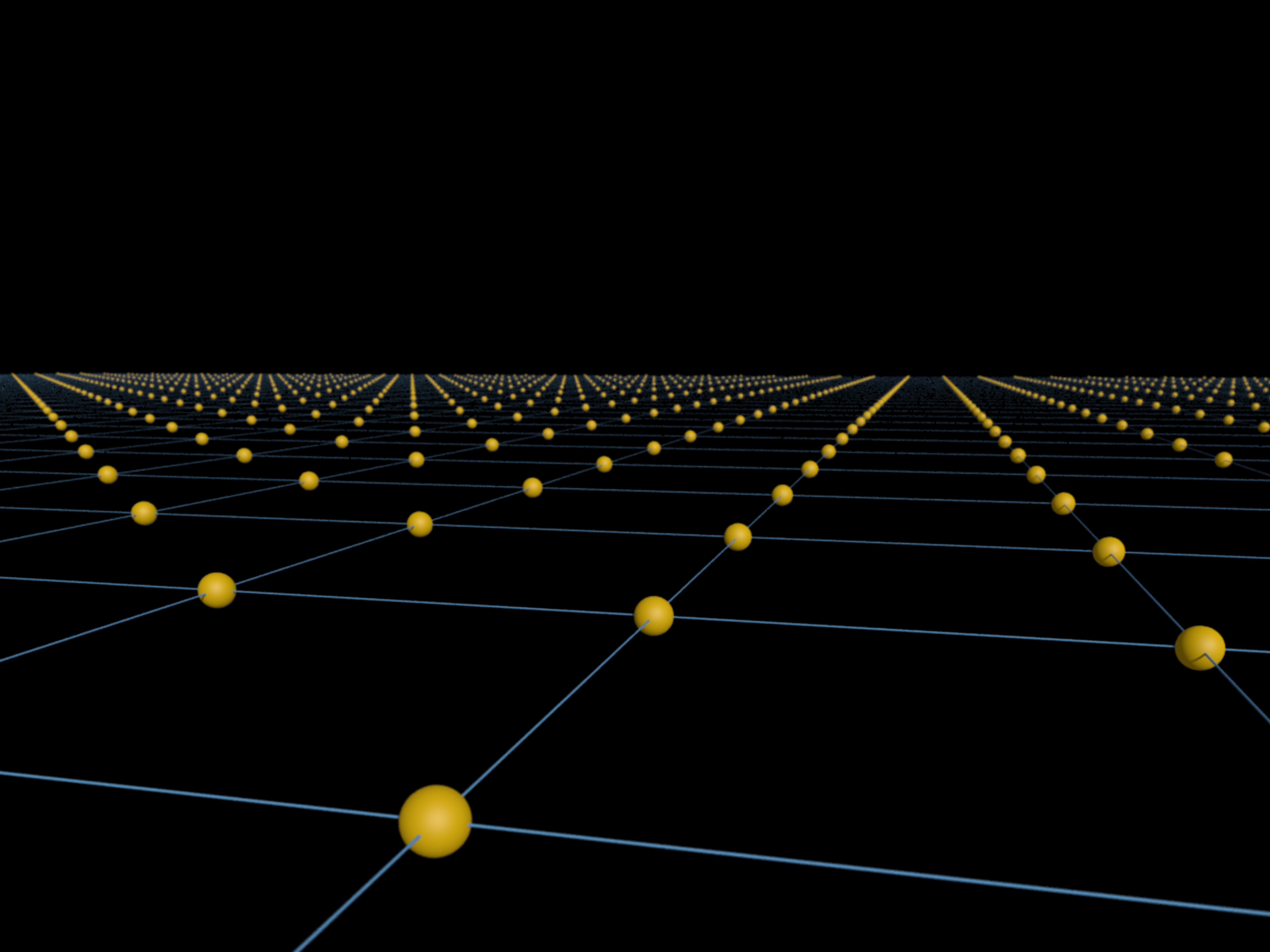
Kitaev, *Ann. Physics* **303**,2003

Ground states

Ground states are *locally indistinguishable*



The thermodynamic limit



Quantum spin systems

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- gives time evolution α_t & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.

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Definition

A state ω on $\mathfrak{A}$ is a *ground state* for H_Λ if we have $-i\omega(A^*\delta(A)) \geq 0$ for all $A \in \delta(A)$. We write *K* for the set of ground states.

Ground states

Suppose we have a state $\omega(A_s) = \omega(B_p) = 1$. Then:

Alicki, Fannes, Horodecki: *J. Phys. A* **40** (2007)
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Lemma

There is a unique state such that $\omega(A_s) = \omega(B_p) = 1$.

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Theorem

There is a unique translation invariant ground state. This ground state is pure and frustration free, and the Hamiltonian in the GNS representation as a spectral gap.

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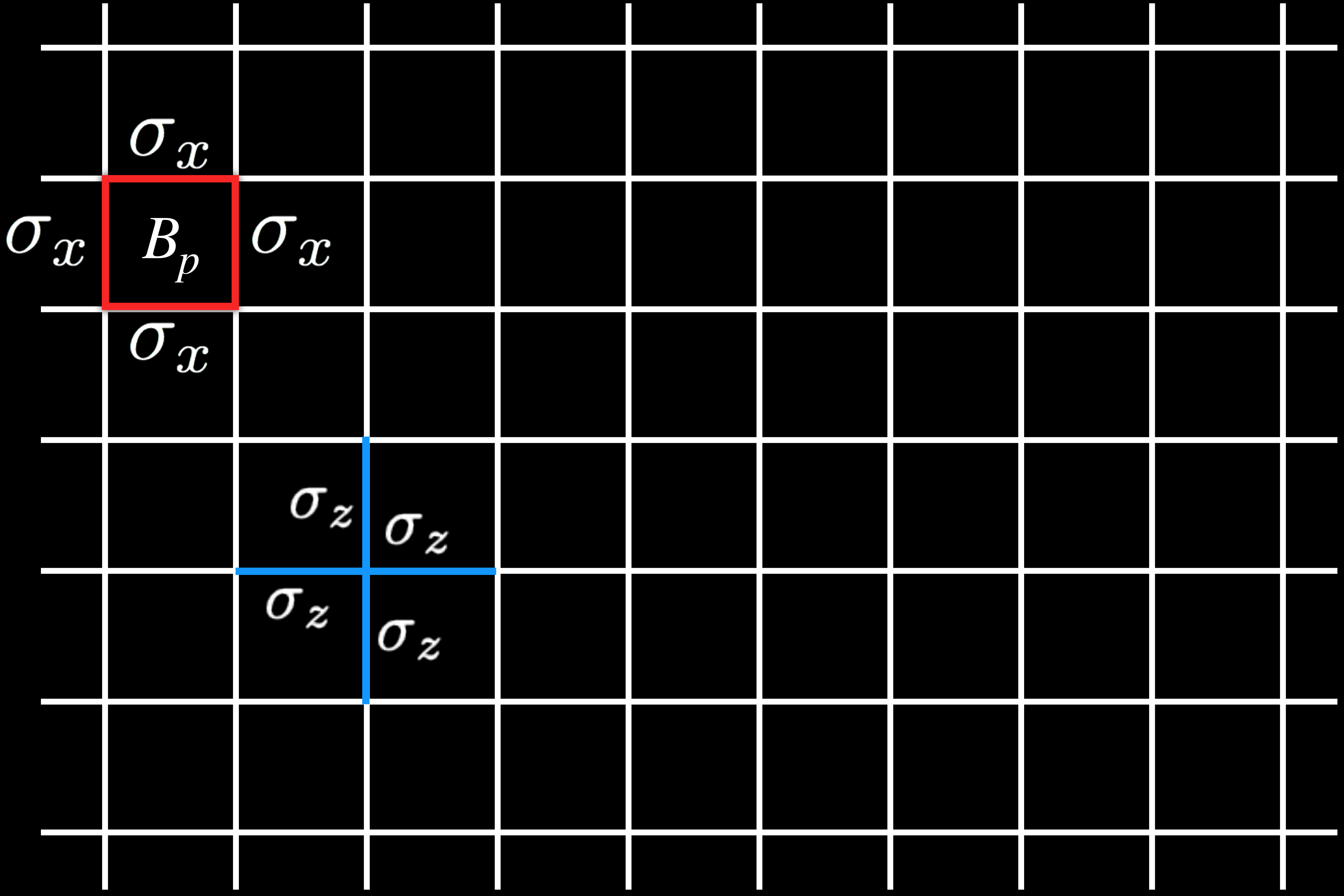
What about non-frustration free states?

Non-frustration free ground states

Toric code: excitations



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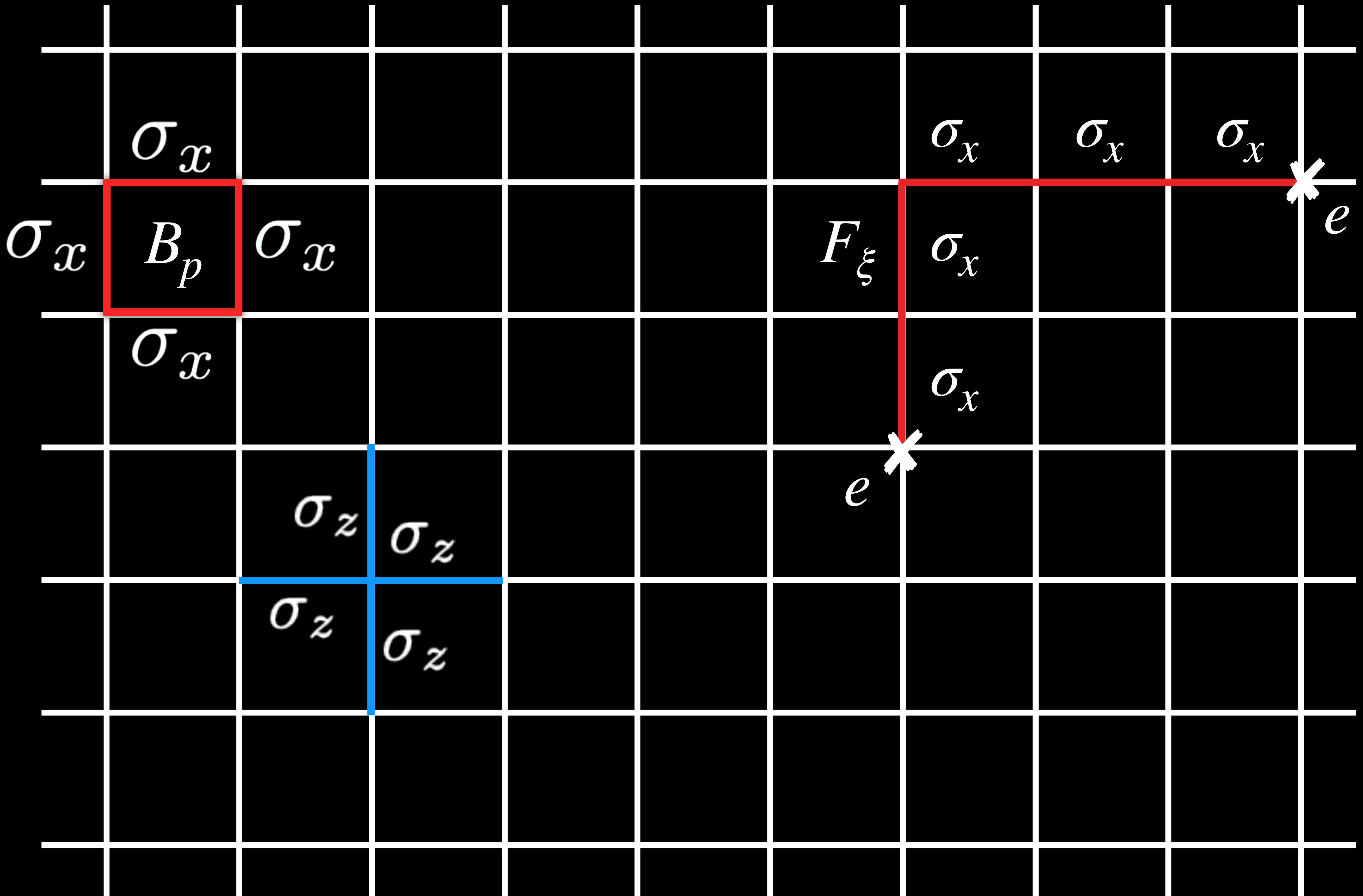

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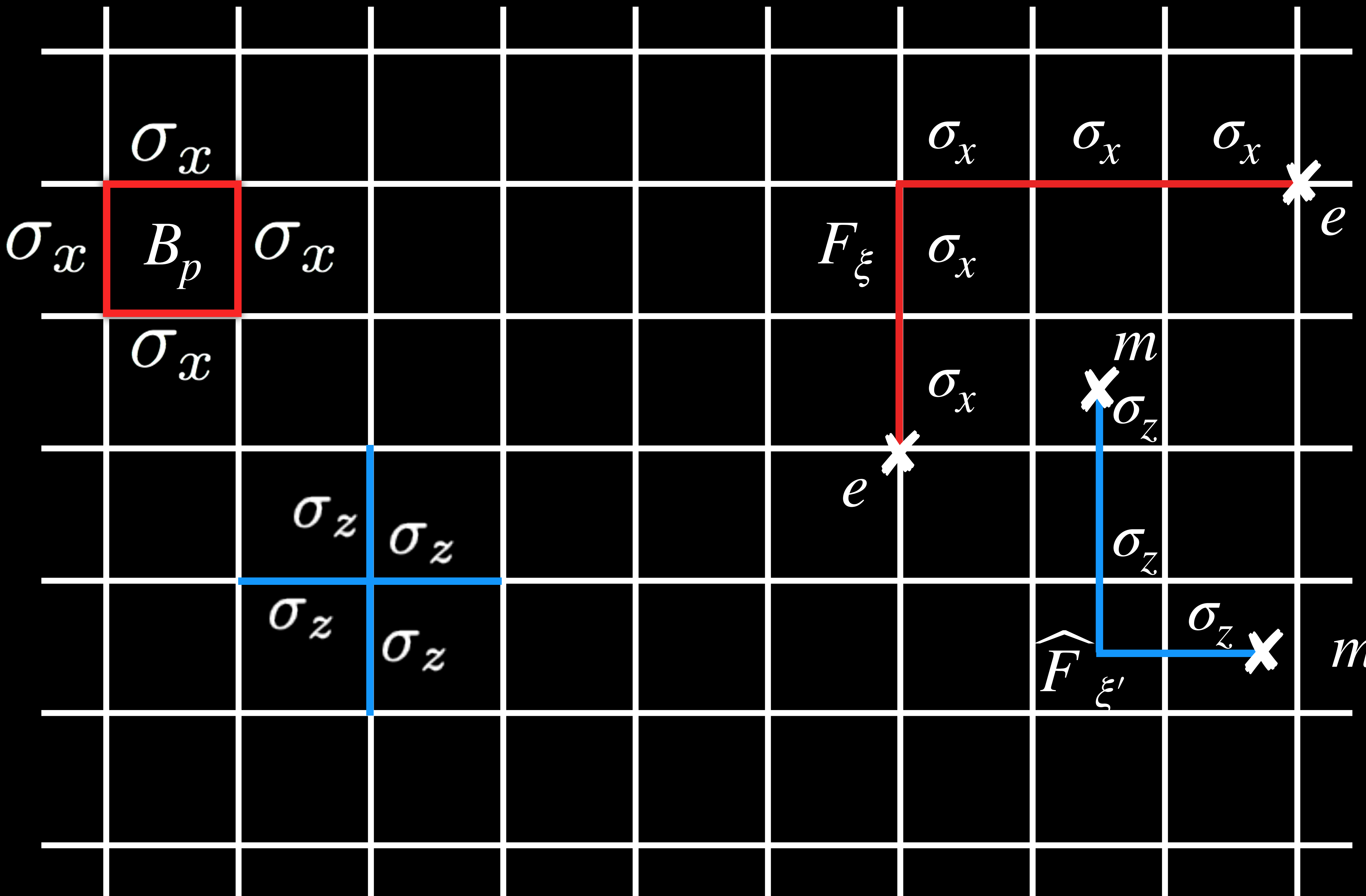

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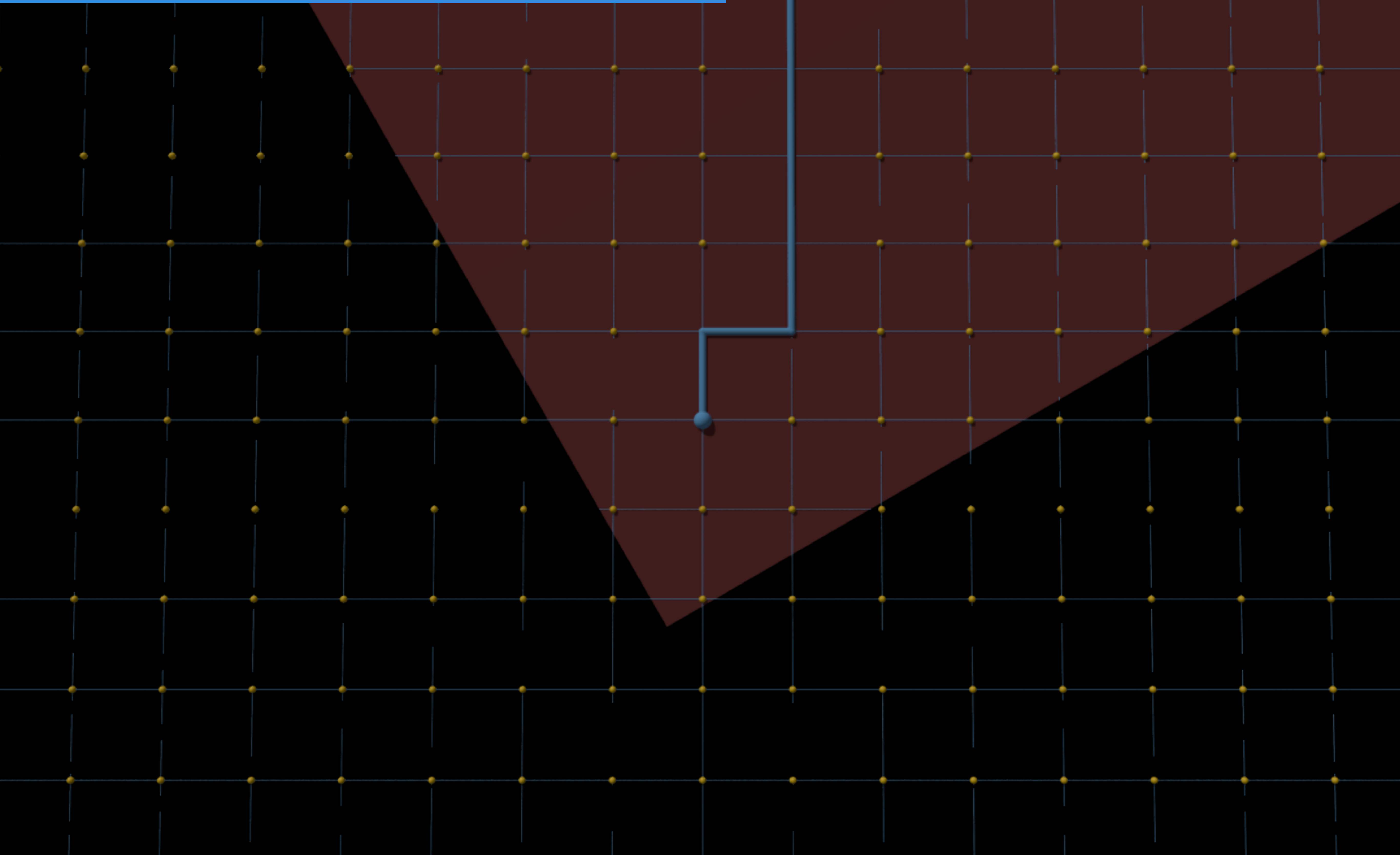
Example: toric code



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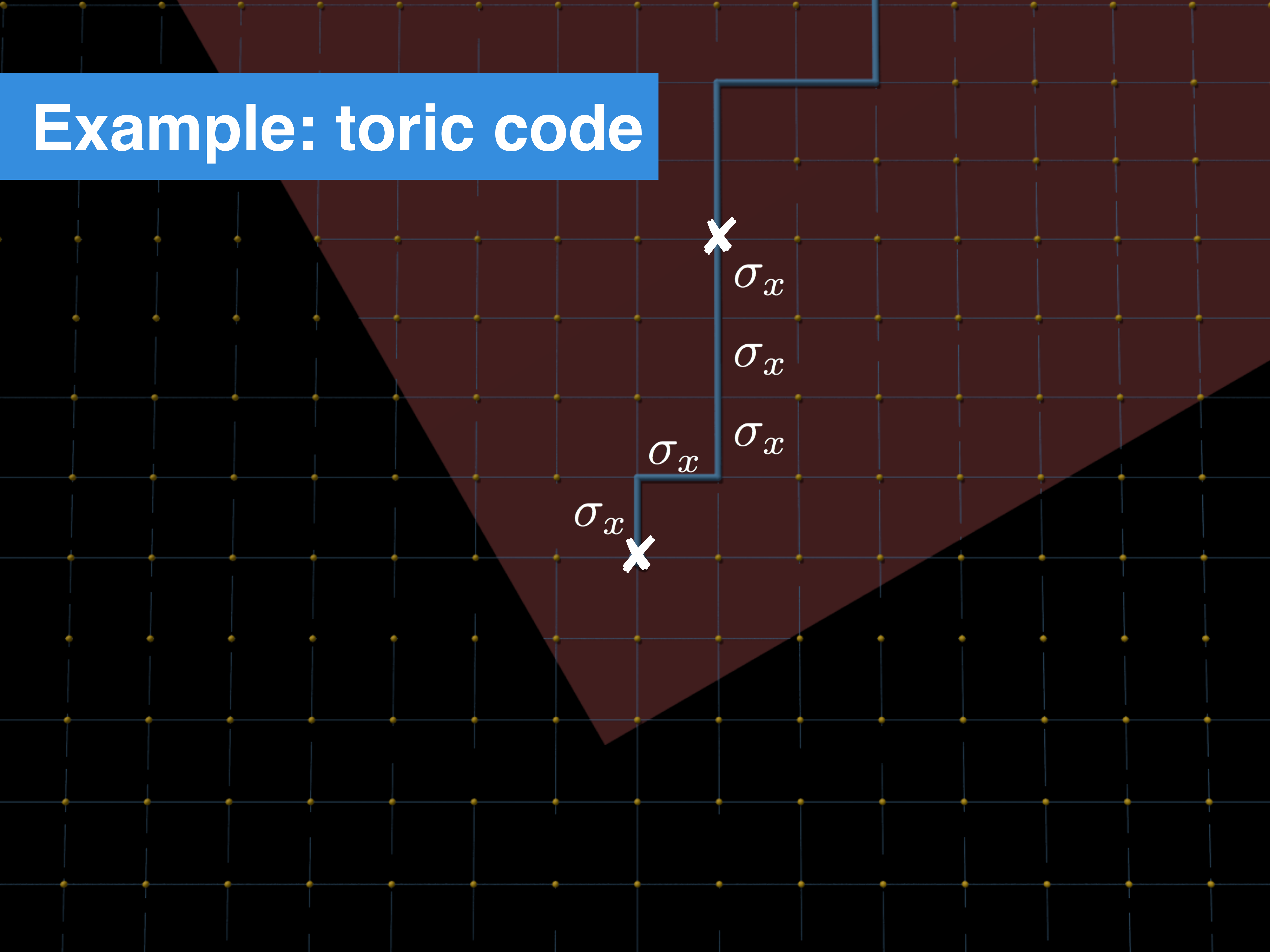

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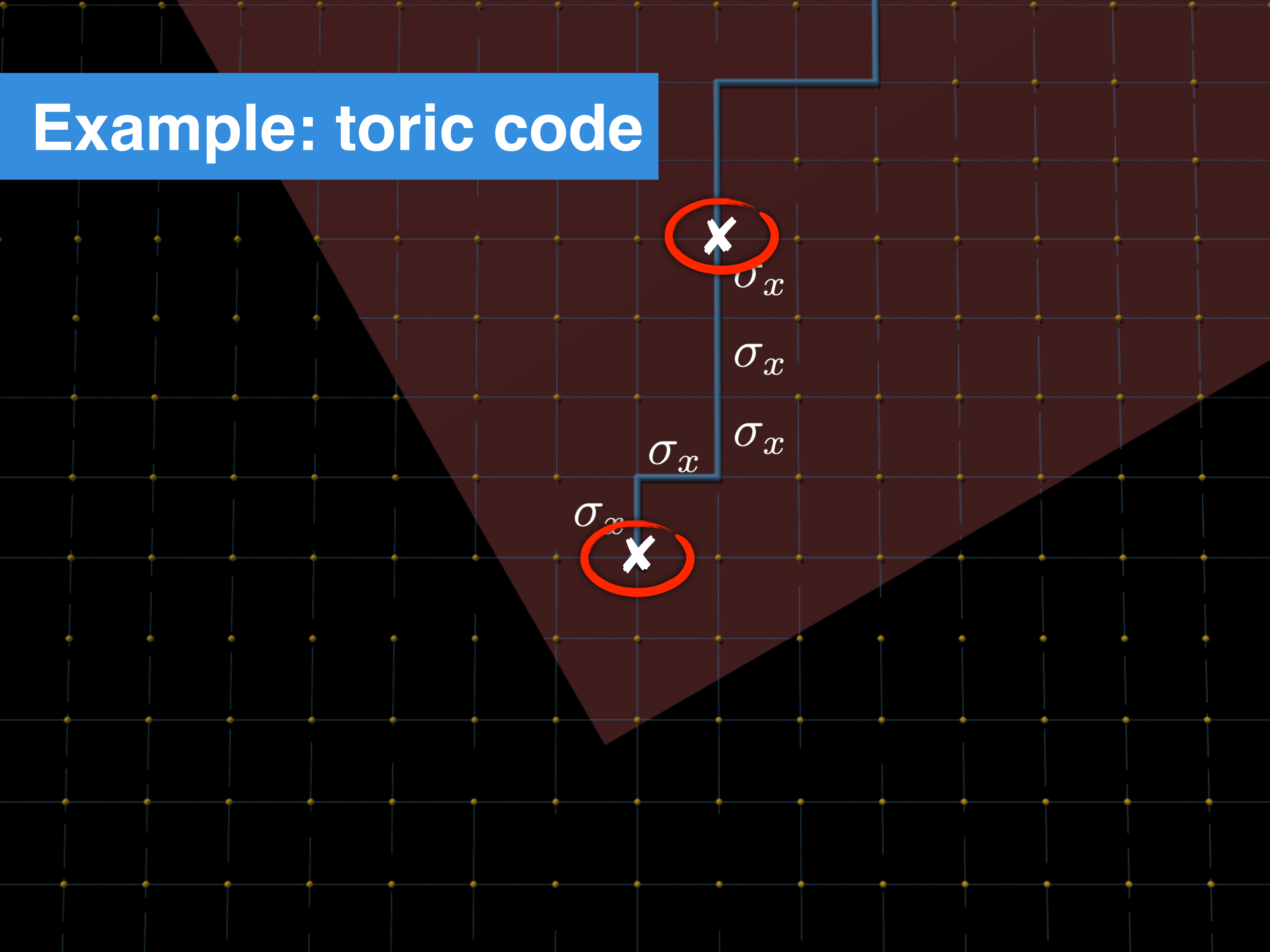

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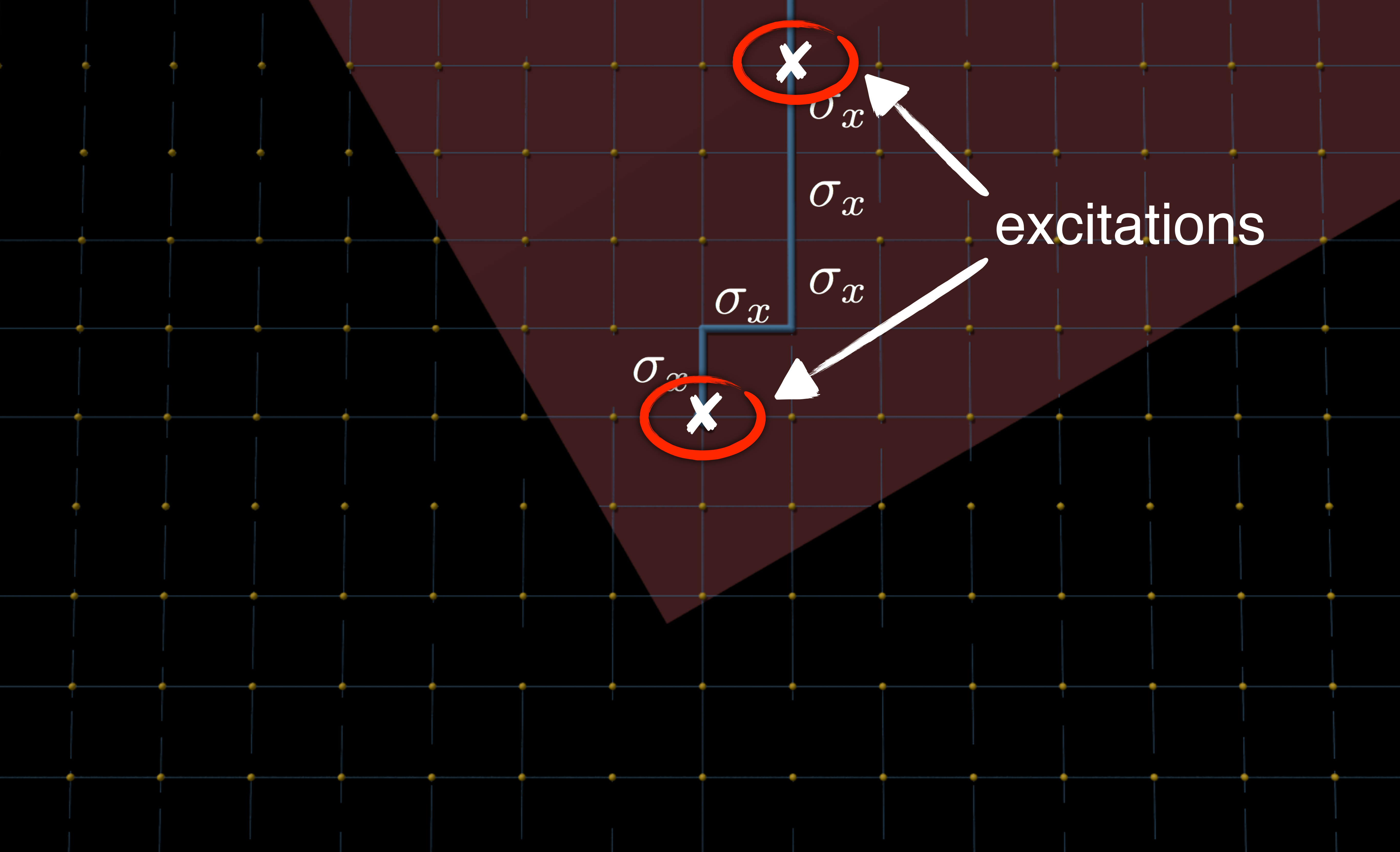
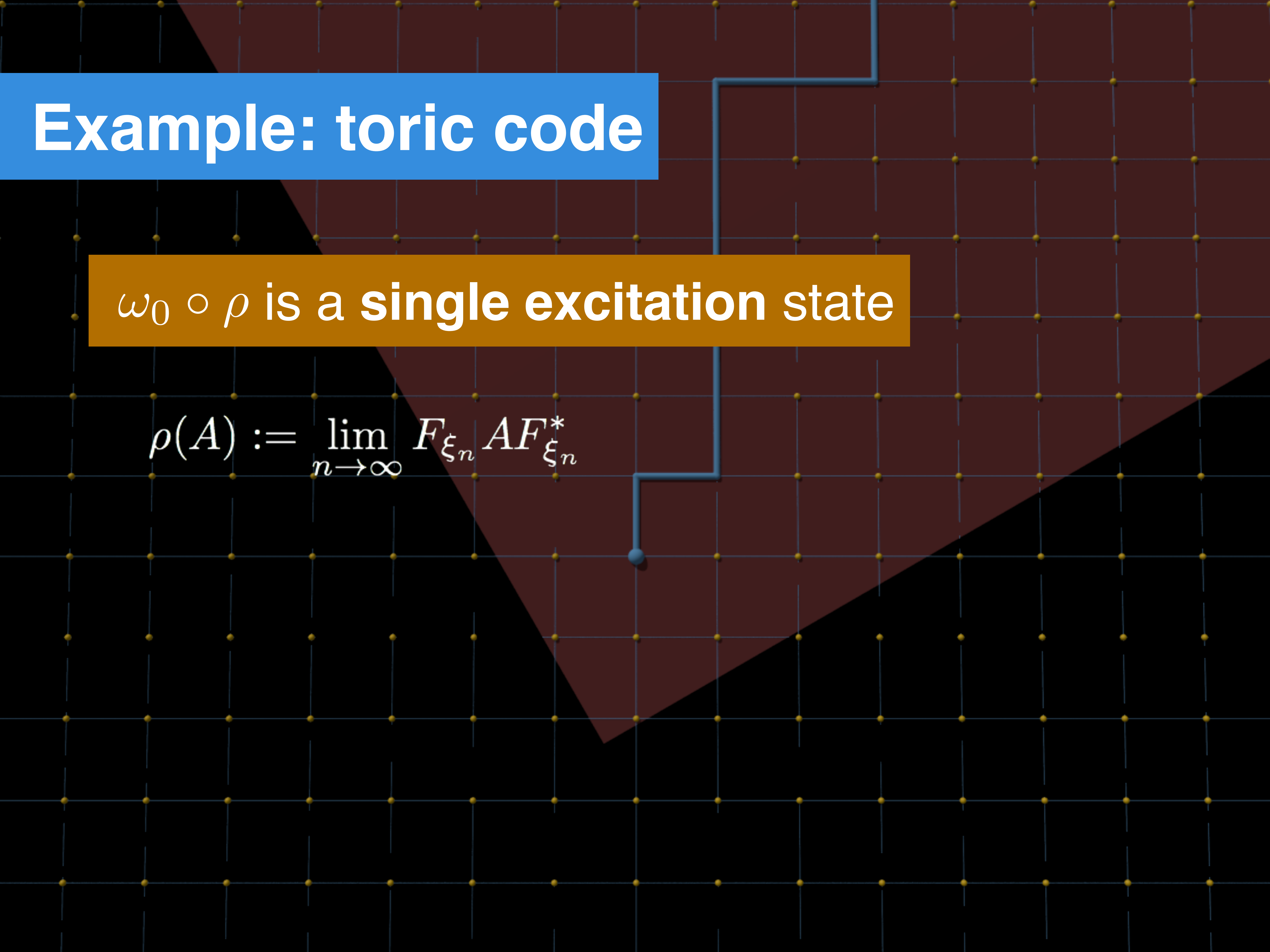


Example: toric code

$\omega_0 \circ \rho$ is a **single excitation** state

$$\rho(A) := \lim_{n\to\infty} F_{\xi_n} A F_{\xi_n}^*$$



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$\pi_0 \circ \rho$ describes observables in presence of **background charge**

Non-frustration free ground states

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The states $\omega \circ \rho$ are ground states!

- $\omega \circ \rho(H_\Lambda) > 0$ whenever Λ contains the endpoint of the string
- Energy can be decreased *locally* (near the excitation) with local operators …
- … but not *globally* (excitation just gets moved around)

Ribbon operators

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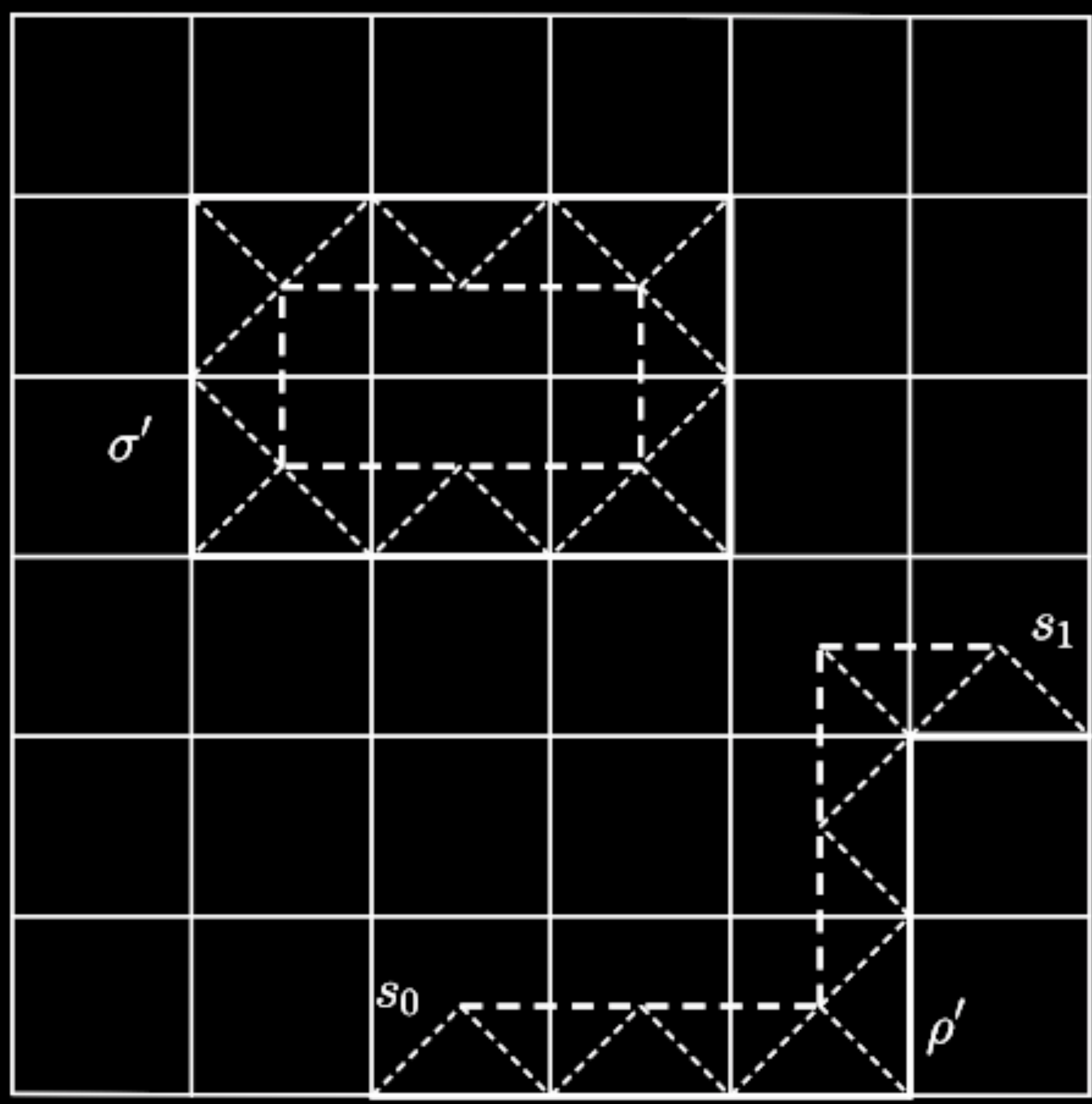

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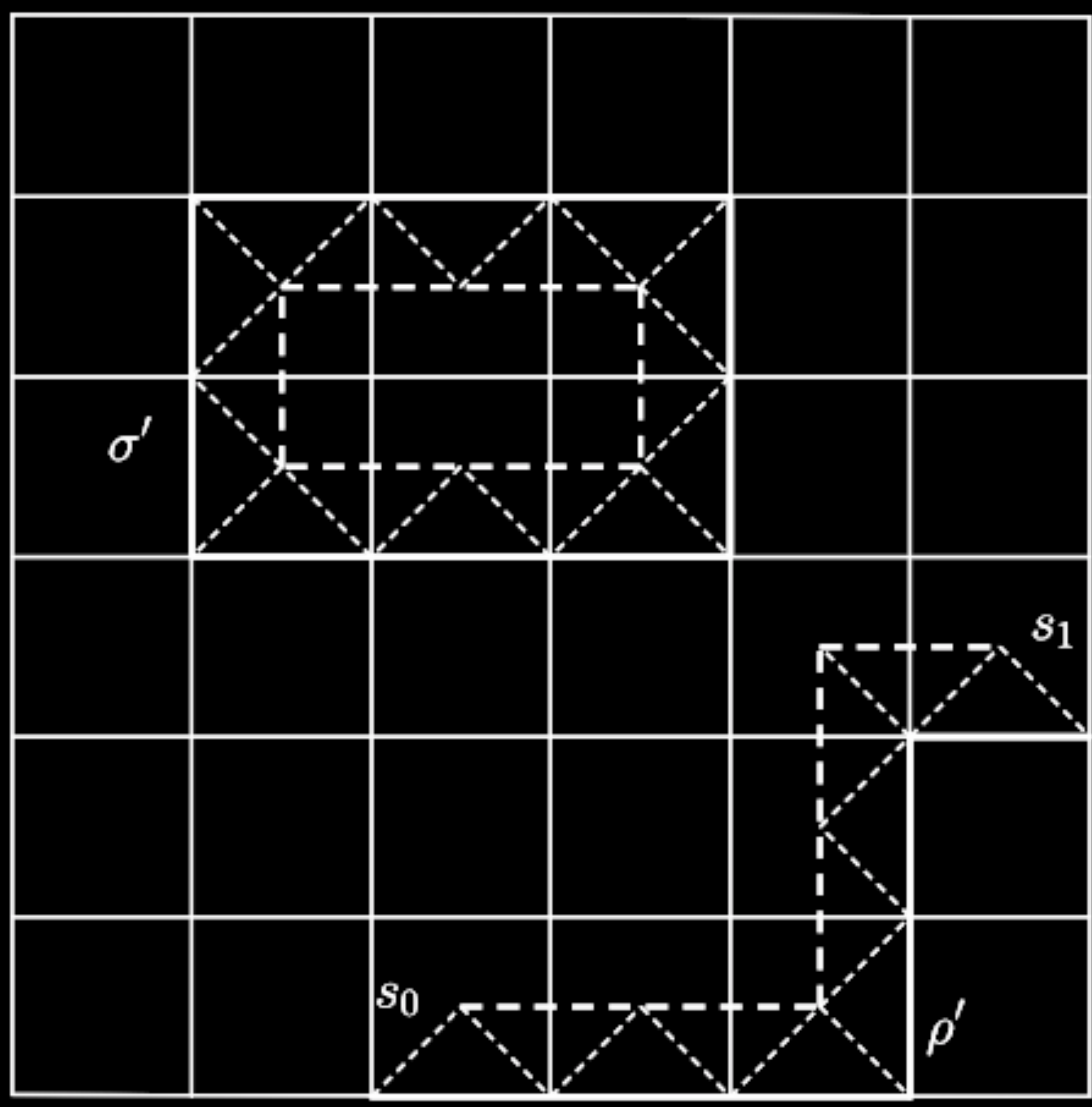

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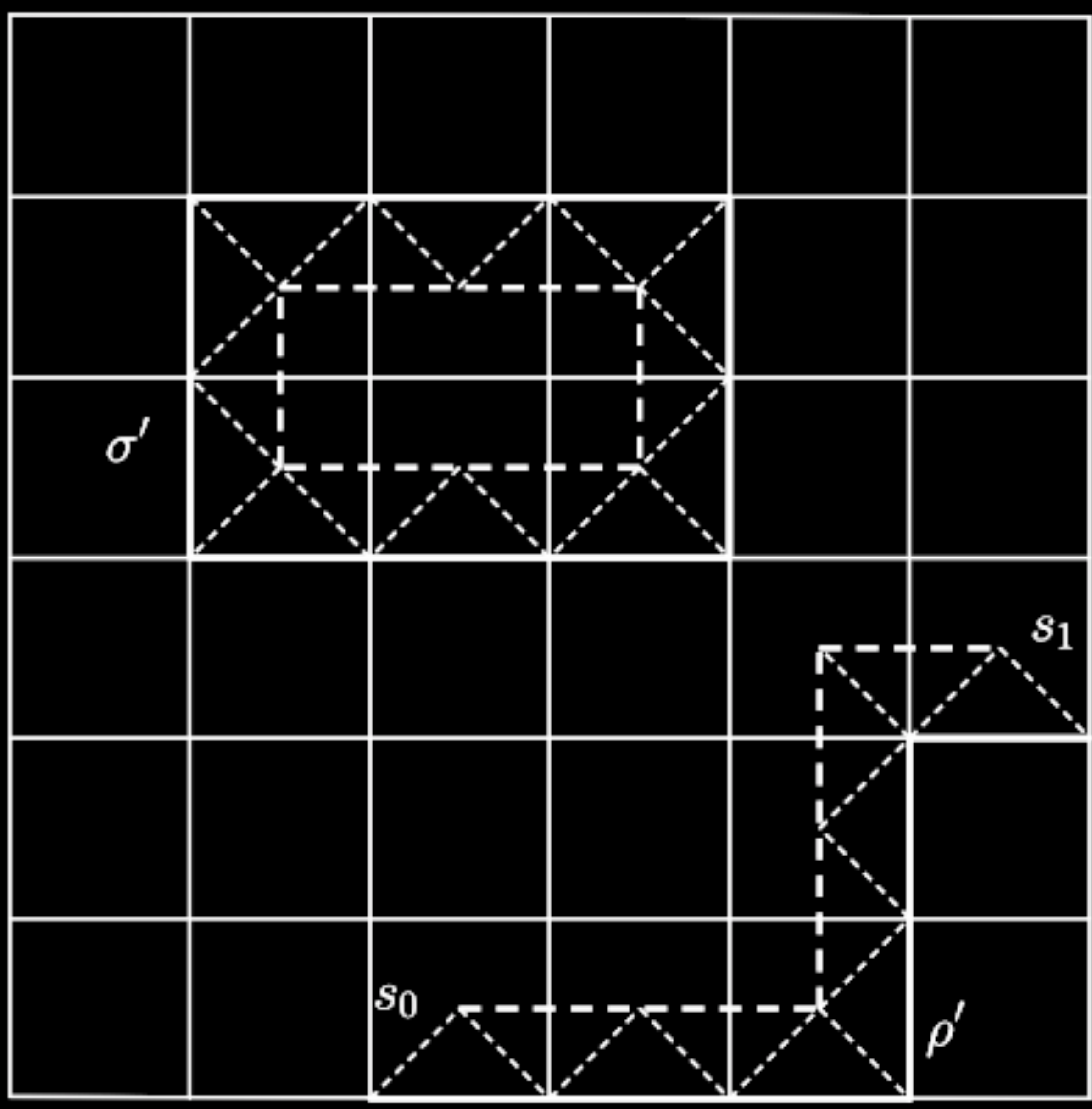

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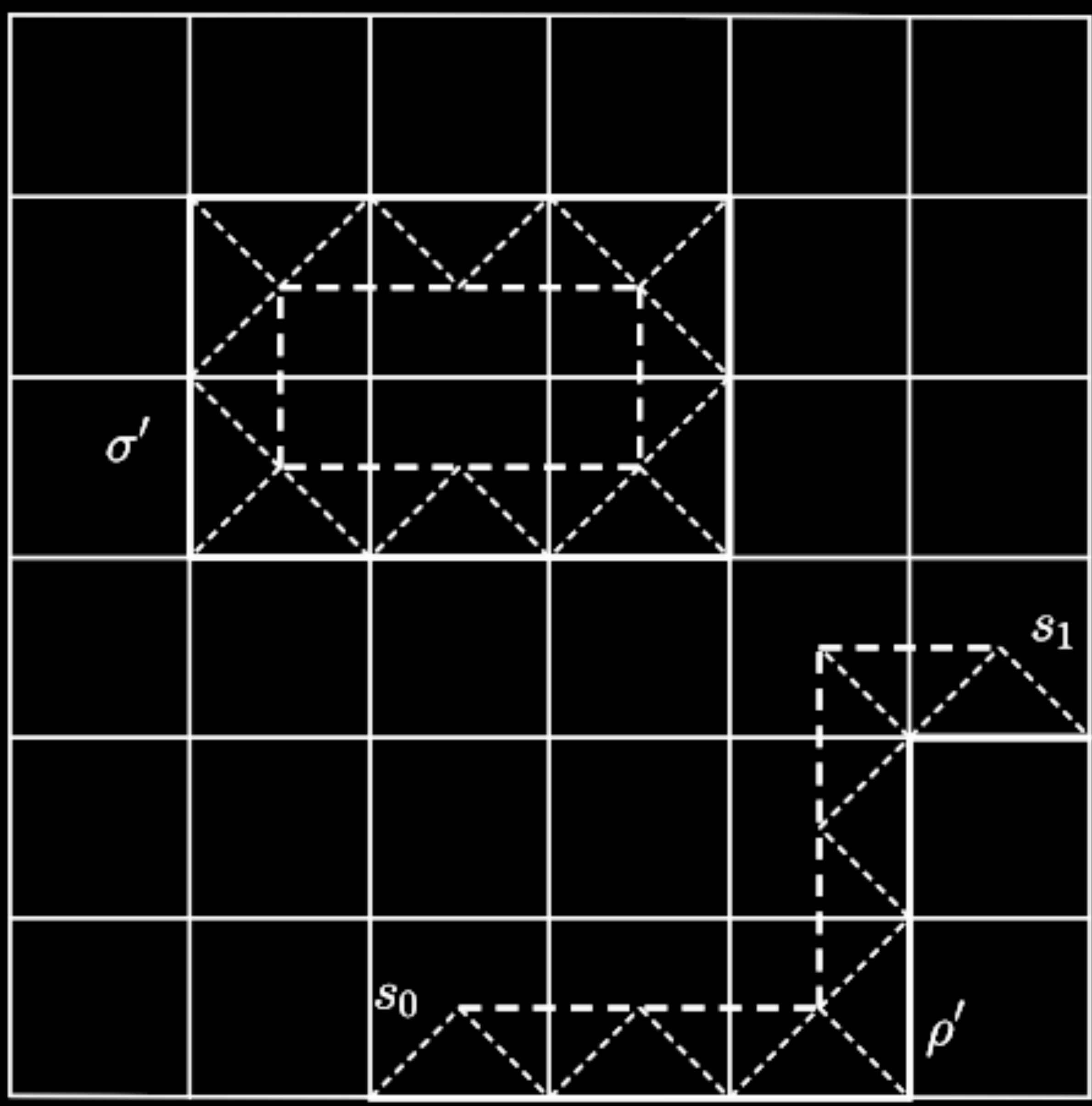

Can define "ribbon operators" $F^{\omega,c}_{\rho'}$

Interpretation: creates charge (χ, c) and conjugate at endpoints

Can define states $\omega_{\chi,c} = \lim_{n\to\infty} \omega_0(F^{\chi,c}_n \cdot (F^{\chi,c}_n)^*)$ as before

Theorem (Cha, PN, Nachtergaele)

Let $\omega \in K$. Then there is a convex decomposition

$$\omega = \sum_{\chi \in \widehat{G}, c \in G} \lambda_{\chi,c}(\omega)\omega^{\chi,c}$$

where $\omega^{\chi,c} \in K^{\chi,c}$. Moreover, each $K^{\chi,c}$ is a face, and every pure state in $K^{\chi,c}$ is unitarily equivalent to the state

$$\omega_{\chi,c} = \lim_{n\to\infty} \omega_0(F_n^{\chi,c} \cdot (F_n^{\chi,c})^*).$$

Commun. Math. Phys. **357**, 125-157 (2018)

Sketch of proof

Boundary terms

Lemma

Let $\widetilde{H}_L \geq 0$ be a sequence of operators such that $\delta(A) = \lim_{L\to\infty} -i[\widetilde{H}_L, A]$ and ω a state such that $\omega(\widetilde{H}_L) = 0$. Then it is a ground state.

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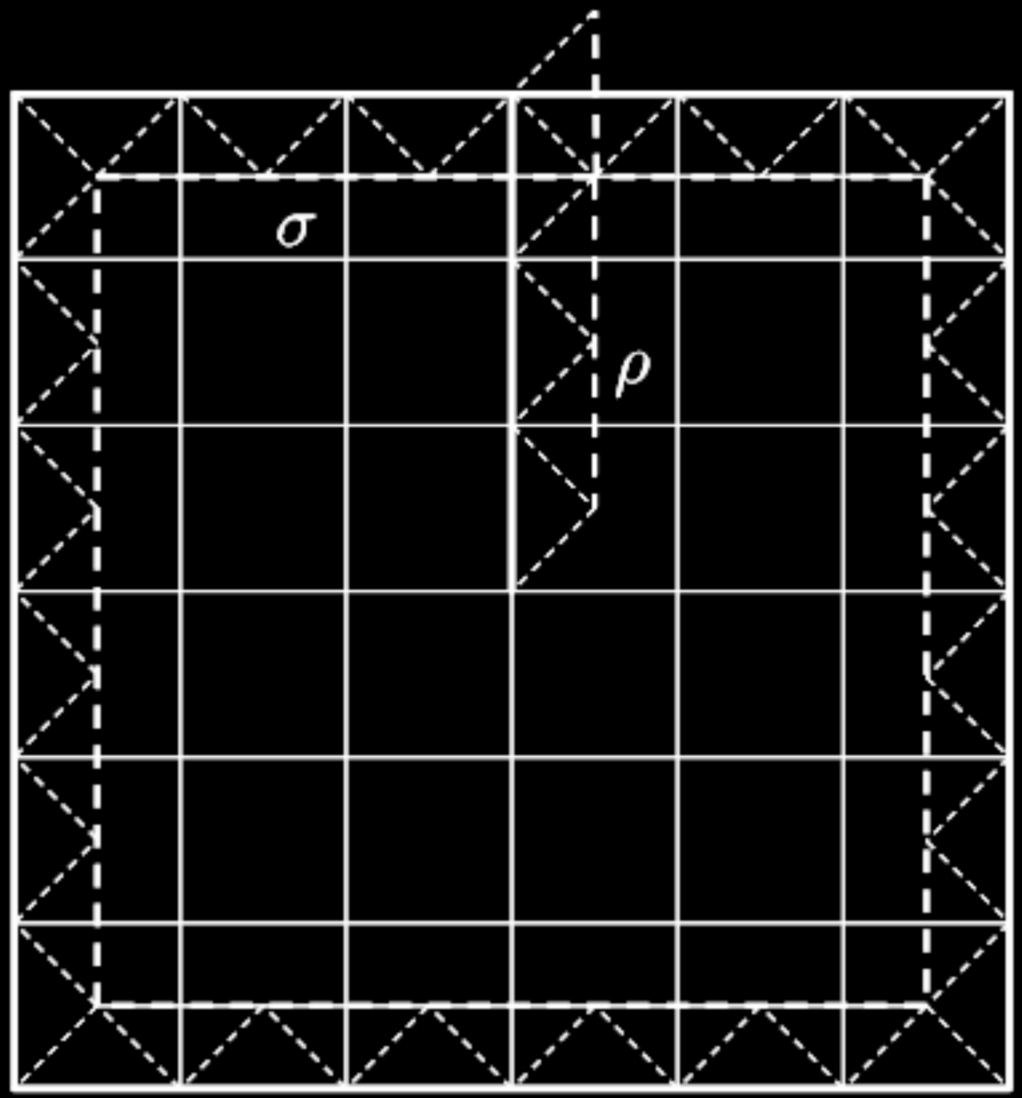
Idea: add suitable boundary terms to quantum double dynamics, and send the boundary to infinity.

Charge measurement



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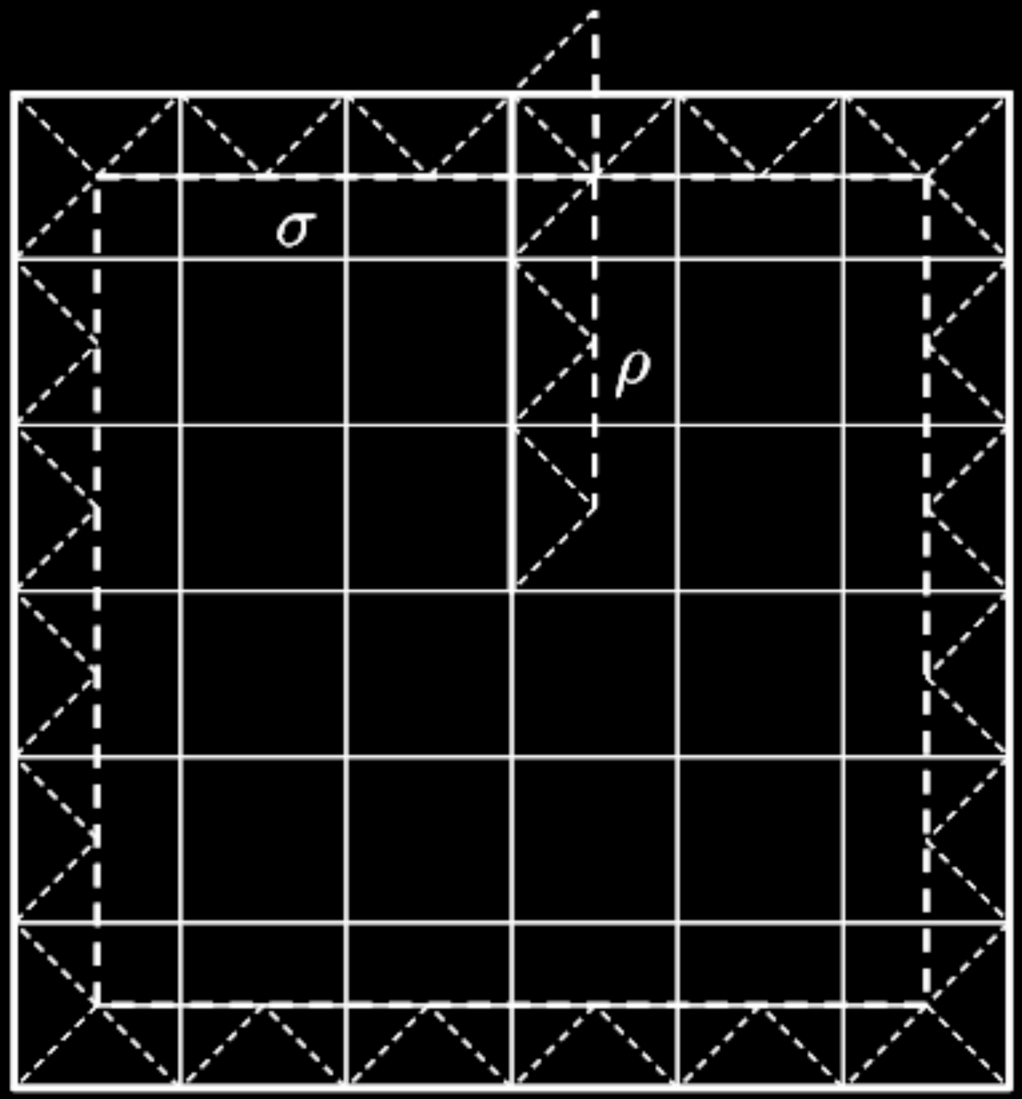

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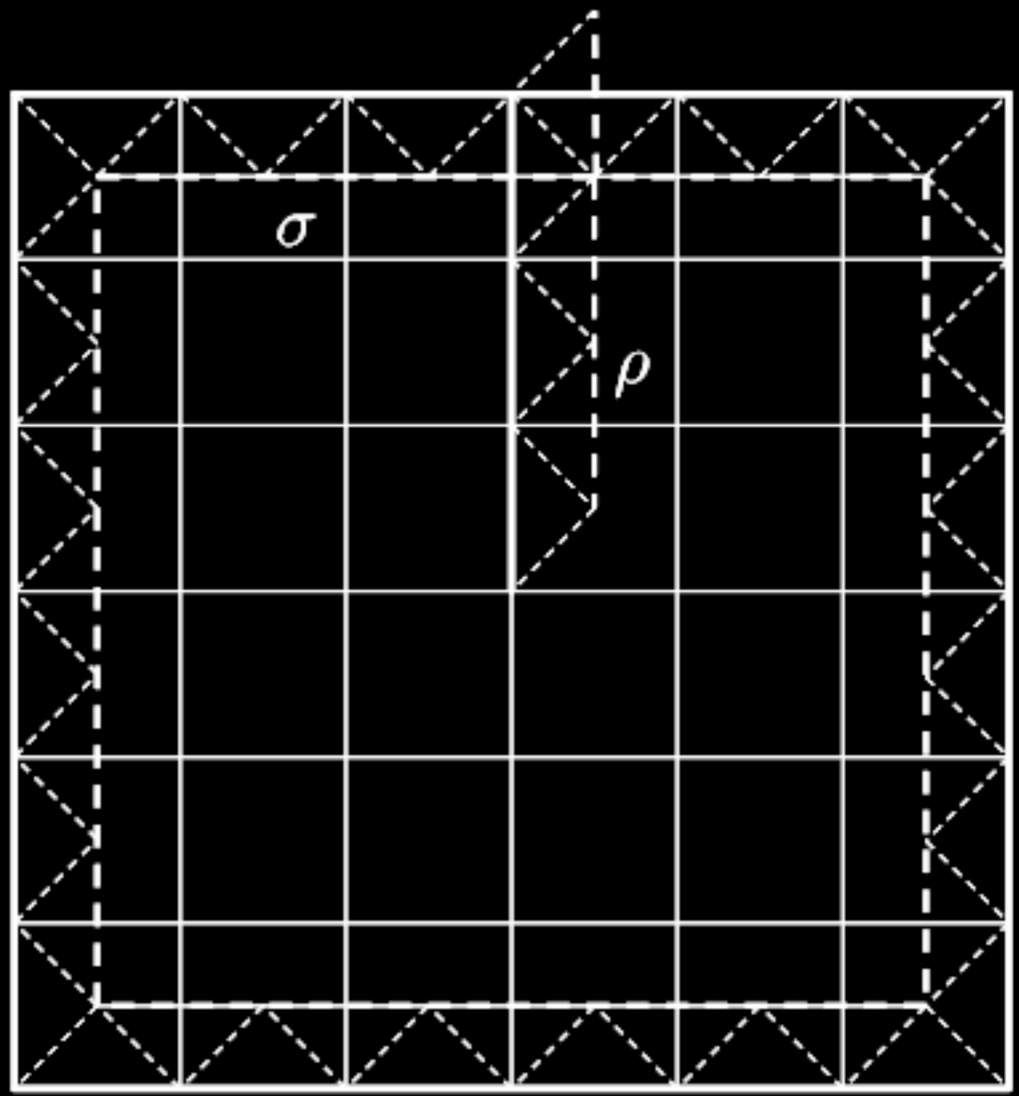

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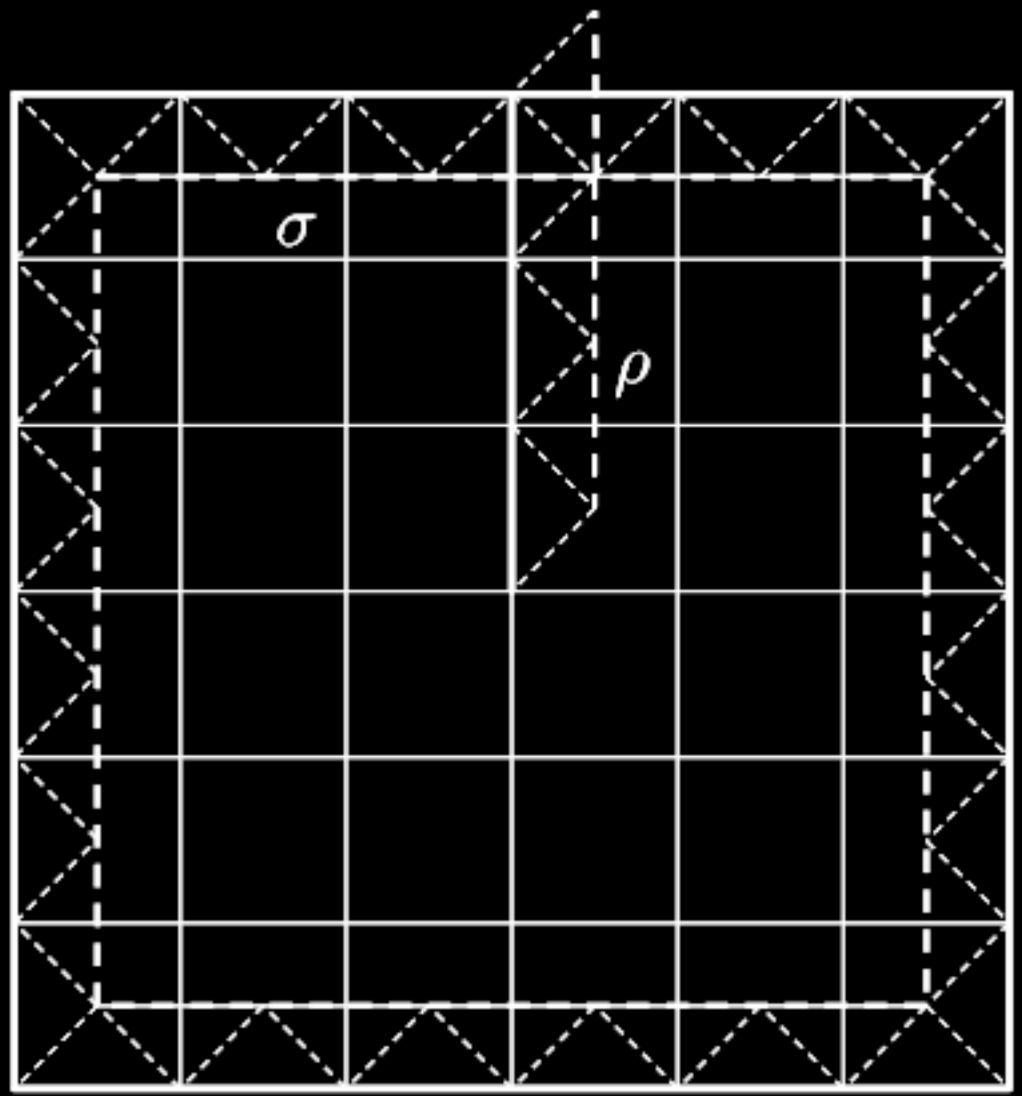

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Using Lemma, it follows that states constructed are ground states

Local charges

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But charges have a group structure!
$(\chi_1, c_1)(\chi_2, c_2) = (\chi_1 \cdot \chi_2, c_1 c_2)$

So how can we define the total charge in a region?

Local charges

$$D_L^{\chi} := \sum_{\prod_i \chi_i = \chi} \prod_i D_{v_i}^{\chi_i} \qquad D_L^{c} := \sum_{\prod_i c_i = c} \prod_i D_{f_i}^{c_i}$$

Lemma

These operators are only supported on the *boundary*. In fact, $F_L^{\chi,c} = D_L^{\chi} D_L^{c}$.

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This allows for an explicit description of local GS

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$$K^{\chi,c} := \left\{ \omega^{\chi,c} : \exists \omega \in K, \lim_{L\to\infty} \omega(D_L^{\chi,c}) > 0 \text{ and} \right.$$

$$\left. \omega^{\chi,c} = w^* - \lim_{L\to\infty} \frac{\omega(\cdot D_L^{\chi,c})}{\omega(D_L^{\chi,c})} \right\}$$

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One can prove that $K^{\chi,c} \subset K$ is a face

Lemma

Let $\omega \in K$. Then the following limit exists:

$$\lambda_{\chi,c}(\omega) = \lim_{L\to\infty} \omega(D_L^{\chi,c}) \geq 0.$$

Furthermore, if $\omega \in K^{\chi,c}$, then we have $\lambda_{\chi',c'}(\omega) = \delta_{c,c'}\delta_{\chi,\chi'}$.

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The non-abelian case

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(Work in progress with Mahdie Hamdan)

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Non-abelian quantum double

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- Closed loops detecting charge
- Local charge projectors (but cannot separate magnetic and electric)
- Composition of charges is more complicated! (cf. irreps of non-abelian groups)

Representations of *D(G)*

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Still have nice (but more complicated) algebraic relations

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The limit converges and gives a localised, unital positive map

Theorem (Mahdie Hamdan, PN)

The states $\omega_0 \circ \chi^{\pi C}$ are ground states of the non-abelian quantum double model. These states are factor states, but not pure (unless the irreducible representation πC is one-dimensional).

Work in progress

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Kitaev & Kong, *Commun. Math. Phys.* **313** (2012)

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In 2D topologically ordered models, excitations in the *bulk* are conjectured to be related to *gapped boundaries*

Our analysis of the ground states can be seen as an implementation of this idea

Kitaev & Kong, *Commun. Math. Phys.* **313** (2012)

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- Satisfied by Kitaev and Levin-Wen models
- Can define a "local" boundary algebra
- Does not refer to Hamiltonians
- Canonical state on bulk and boundary

Theorem

For Levin-Wen models, the canonical state on the boundary is a KMS-1 state, and the corresponding von Neumann algebra is a factor of Type II_1 or Type III.

C. Jones, PN, D. Penneys & D. Wallick, arXiv:2307.12552

Theorem

For Levin-Wen models, the canonical state on the boundary is a KMS-1 state, and the corresponding von Neumann algebra is a factor of Type II_1 or Type III.

Theorem

The category of “DHR bimodules” of the boundary net is the Drinfeld centre $Z(\mathscr{C})$, where $\mathscr{C}$ is the input fusion category for the Levin-Wen model.

C. Jones, PN, D. Penneys & D. Wallick, arXiv:2307.12552

Theorem

Let ω be the frustration free ground state of the quantum double model for an abelian group G. Then for any convex cone Λ, the von Neumann algebra $\pi_\omega(\mathfrak{A}(\Lambda))''$ is a factor of Type II_∞.

Y. Ogata, arXiv:2212.09036
C. Jones, PN, D. Penneys & D. Wallick, arXiv:2307.12552