Introduction to superselection sector theory I

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QFT and Topological Phases via Homotopy Theory and Operator Algebras

30 June – 3 July, 2025

Outline

1. Introduction

2. The toric code

3. Excitations

Introduction

What to expect?

Main issue: the classification of gapped ground states:

- We will always work in the thermodynamic limit
- Gapped ground states of local Hamiltonians…
- … with some equivalence relation
- Focus on states with topological order (or long-range entanglement)
- Non-invertible states

Question

Can we find (physically interesting) invariants?

Why are these states interesting?

- Can host anyons: quasi-particles/superselection sectors/charges/... with braided statistics
- Algebraic properties of anyons are described by braided tensor C*-categories (typically even modular or braided fusion)
- ‘Topological’ nature makes these properties robust
- In other words, the category should be an invariant

Question

How can we obtain the category of anyons from a microscopic description of the state? (And is this indeed an invariant?)

What *not* to expect?

- Not a historical overview
- Only non-chiral topological order
- Will focus on basics, not most general statements
- Only discuss the operator-algebraic “DHR approach” to superselection sectors
- Will gloss over more technical details

Plan for the week

- Lecture 1: The toric code and its ground states
- Lecture 2&3: The category of superselection sectors
- Lecture 4: Classification of phases and long-range entanglement

We illustrate the theory by the example of the toric code, but methods work much more general!

Some history

Approach is rooted in Doplicher-Haag-Roberts theory:[1]

- Originates in algebraic quantum field theory, defined in terms of Haag-Kastler nets of observables $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$
- DHR theory attempts to capture 'charges' and leads to Bose/Fermi (para-)statistics in (3+1)D
- Culminates in Doplicher-Roberts theorem: a STC*-category is equivalent to Rep G for some compact group G
- In lower dimensions, can get braided statistics (anyons!)
- Similar techniques have been very successful in CFT (conformal nets)

[1] Haag, *Local Quantum Physics*, Springer (1992)

Different approaches

The main feature of the approach is the appearance of a braiding (describing anyon exchange).

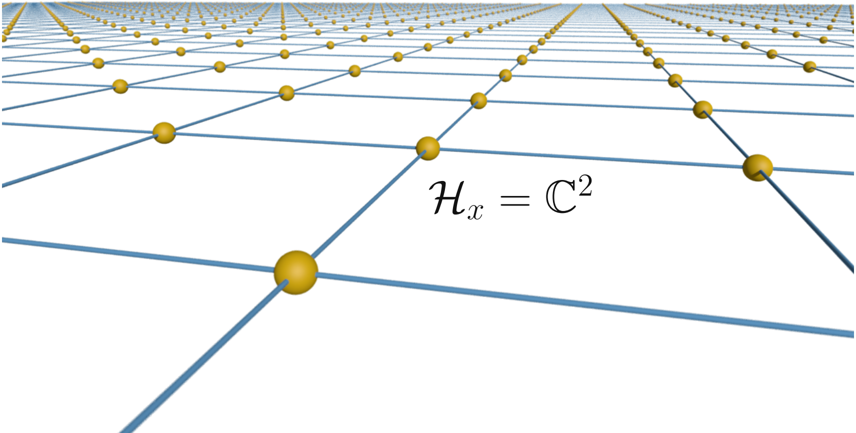
Question

How does this braiding appear?

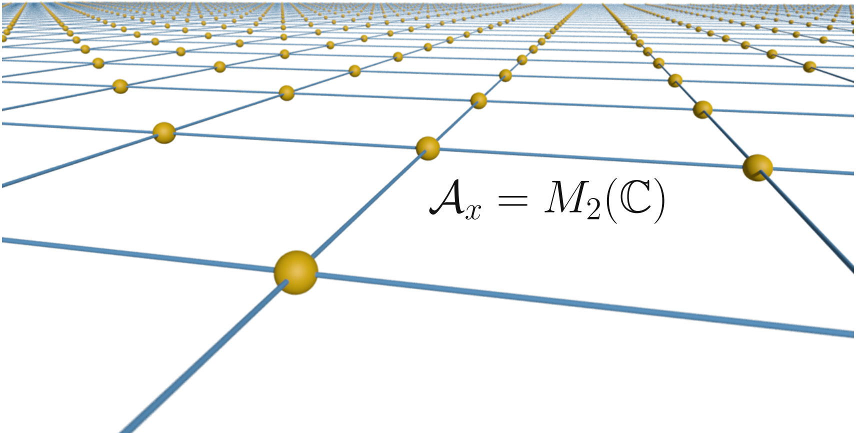
- 'Classical DHR approach': these lectures
  (See also Ogata, arXiv:2106.15741)
- Prefactorisation algebras (geometric approach)
  (Benini, Carmona, PN, Schenkel, arXiv:2505.07960)
  → talk Alexander Schenkel next week
- Axiomatic approach: nets on certain posets
  (Bhardwaj, Brisky, Chuah, Kawagoe, Keslin, Penneys, Wallick: arXiv:2410.21454)

The toric code

The toric code



The toric code



Pauli matrices

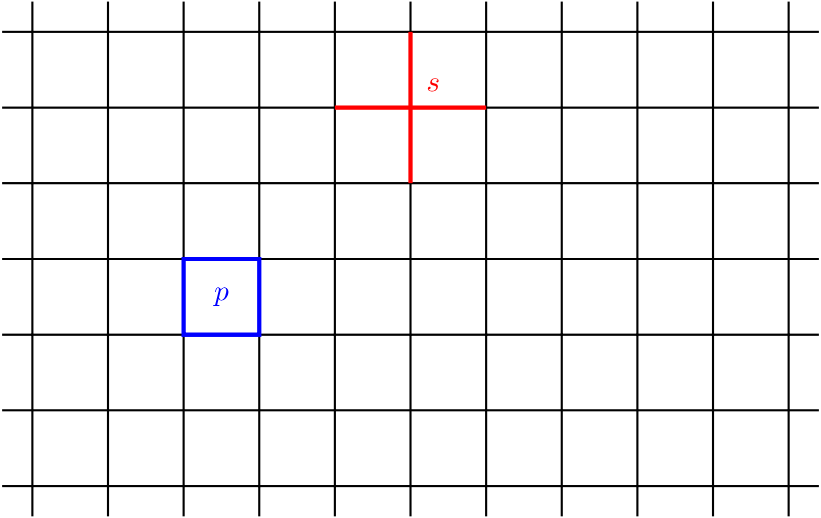
Recall the definition of the Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They have nice algebraic properties: $\{\sigma^i, \sigma^j\} = 2\delta_{i,j} I$:

- Square to the identity
- Different Pauli matrices anti-commute
- Together with I form a basis of $M_2(\mathbb{C})$.

Stars and plaquettes



Dynamics

We define star and plaquette operators:

$$A_s = \bigotimes_{j \in s} \sigma_j^x, \qquad B_p = \bigotimes_{j \in p} \sigma_j^z.$$

Some easy properties: $A_s^2 = B_p^2 = I$, and all commute.
Can use this to define the dynamics:

$$H_\Lambda = \sum_{s \subset \Lambda} (I - A_s) + \sum_{p \subset \Lambda} (I - B_p)$$

Note that the dynamics are very simple ("commuting projector")!

Frustration-free ground state

Lemma

Let $X_i \leq I$ be a set of operators and suppose that there is a unique state ω such that $\omega(X_i) = 1$ for all X_i. Then ω is pure.

Proof.

Let ϕ be a positive linear functional such that $\phi \leq \omega$. Since $I - X_i \geq 0$, we have

$$0 \leq \phi(I - X_i) \leq \omega(I - X_i) = 0.$$

Hence $\phi(X_i) = \phi(I)$. From the uniqueness assumption, $\phi = \phi(I)\omega$, and it follows that ω is pure. $\square$

Lemma

Let $X \leq I$ with $\omega(X) = 1$. Then $\omega(A) = \omega(AX) = \omega(XA)$.

Frustration-free ground state

Theorem

The toric code has a unique frustration free ground state ω_0. This state is pure.

Proof.

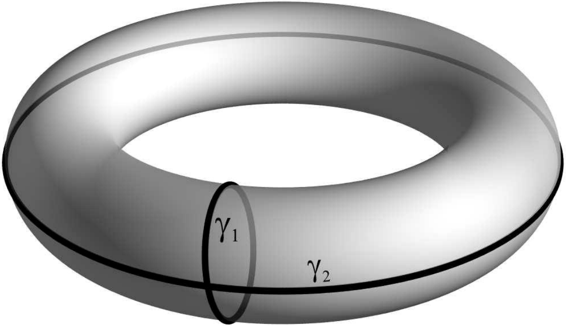
One can show (exercise!) that there is a state such that $\omega_0(A_s) = \omega_0(B_p) = 1$ for all star and plaquette operators, and these conditions uniquely determine it. Hence ω_0 is pure. Note that $\omega_0(I - A_s) = \omega_0(I - B_p) = 0$. For $A \in \mathfrak{A}_{\text{loc}}$, we have

$$\begin{aligned} -i\omega_0(A^*\delta(A)) &= \sum_s \omega_0(A^*AA_s) - \omega_0(A^*A_sA) + B_p \textbf{ terms} \\ &= \sum_s \omega_0(A^*(I - A_s)A) + B_p \textbf{ terms} \\ &\geq 0 \end{aligned}$$

□

An aside...

When defined on a non-trivial topology (e.g. a torus), the condition $\omega(A_s) = \omega(B_p) = 1$ fixes the state locally but not globally. In fact, the ground state space is a quantum error correction code!



Ground space degeneracy is given by 4^g

GNS representation

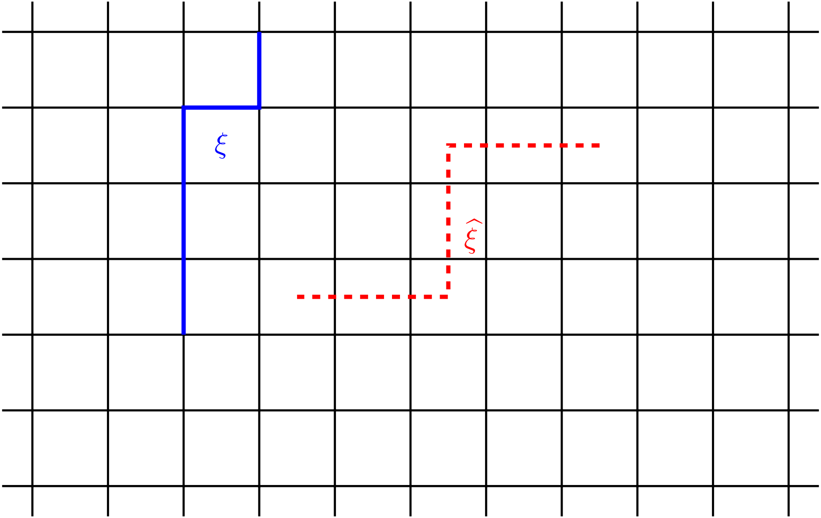
We will use ω_0 throughout as a reference state.

- GNS representation $(\pi_0, \Omega, \mathcal{H}_0)$
  - $\pi_0 : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ $*$-representation
  - $\pi_0(\mathfrak{A})\Omega$ dense in $\mathcal{H}_0$
  - $\omega_0(A) = \langle \Omega, \pi_0(A)\Omega \rangle_{\mathcal{H}_0}$
- Will often identify $\pi_0(A)$ with A
- We have $A_s\Omega = B_p\Omega = \Omega$ (stabiliser condition):
  $\rightsquigarrow H_\Lambda \Omega = 0$ for all Λ
- Hamiltonian in GNS representation with $H\Omega = 0$, $H \geq 0$ satisfies $\mathbf{spec}(H) \cap (0, 2) = \emptyset$ (spectral gap)
- State satisfies LTQO conditions: spectral gap is stable!

Excitations

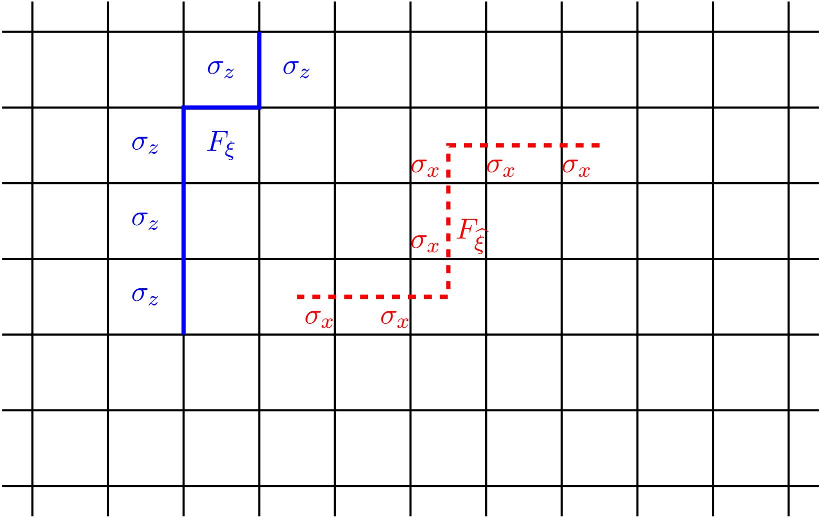
Path operators

We can consider paths ξ and dual paths $\widehat{\xi}$:



Path operators

And corresponding operators F_ξ and $F_{\widehat{\xi}}$:



Path operators

The edges on which the path operators act always have an even number in common with star and plaquette operators and hence

$$[F_\xi, A_s] = [F_\xi, B_p] = [F_{\hat{\xi}}, A_s] = [F_{\hat{\xi}}, B_p] = 0$$

except at the endpoints of the path! Path operators F_ξ anti-commute with star operators at endpoint, whilst the $F_{\hat{\xi}}$ anti-commute with the plaquette operators.

Excitations

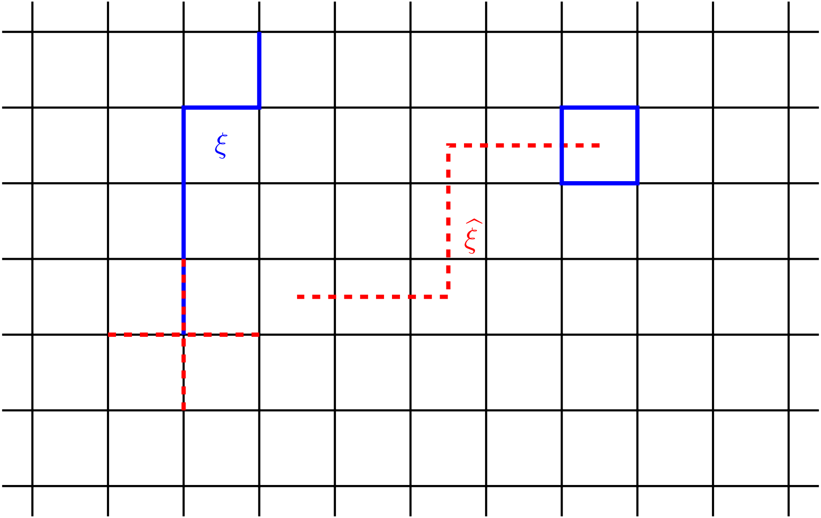
The path operators create a pair of electric or magnetic excitations respectively.

We have

$$H_\Lambda F_\xi \Omega = 2\#(\partial\xi \cap \Lambda) F_\xi \Omega, \qquad H_\Lambda F_{\hat{\xi}} \Omega = 2\#(\partial\hat{\xi} \cap \Lambda) F_{\hat{\xi}} \Omega$$

where $\#(\partial\xi \cap \Lambda)$ is the number of endpoints of ξ within Λ

Path operators



Single excitations

The state $F_\xi \Omega$ describes a pair of anyons/excitations. Alternatively, in the Heisenberg picture,

$$\rho_\xi^Z(a) := F_\xi a F_\xi^* = (\mathbf{Ad}\, F_\xi)(a)$$

is an automorphism of $\mathfrak{A}$ that describes how observables change in the presence of the two excitations.

Question

Can we describe a *single* excitation?

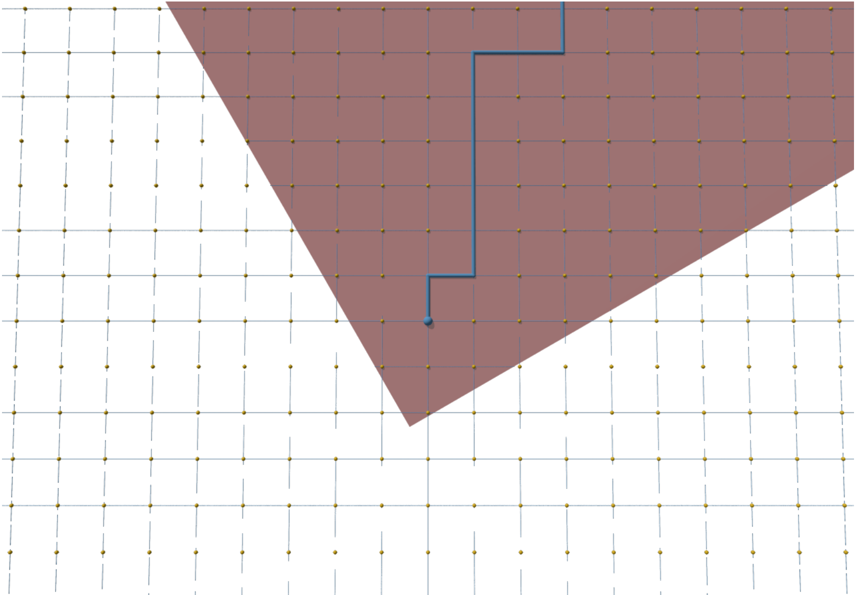
Answer

Yes! We work on an infinite lattice, so can send one of the excitations to infinity:

$$\rho_\xi^Z(a) := \lim_{n\to\infty} F_{\xi_n} a F_{\xi_n}^*$$

where ξ_n are the first n parts of a semi-infinite ribbon ξ.

Cones



Localised automorphisms

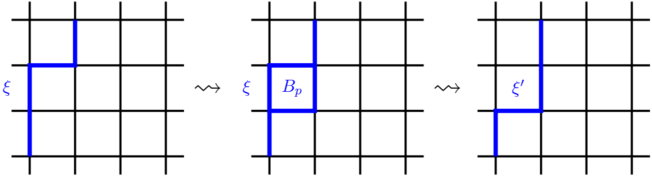
- Can choose a cone Λ as in the picture...
- ... and a semi-infinite path $\xi \subset \Lambda$.
- We get a corresponding automorphism ρ_ξ^Z.
- This is localised in Λ: $\rho_\xi^Z(a) = a$ for all $a \in \mathfrak{A}(\Lambda^c)$.

Notation

We can do something similar for dual paths to get ρ_ξ^X or for the combination of a path and dual path, to get ρ_ξ^Y. We will use the notation ξ for all types of paths, with notation ρ_ξ^k for $k = X, Y, Z$. By definition, $\rho_\xi^0 = \mathbf{id}$.

Single anyon states

The states $\omega_0 \circ \rho_\xi^k$ describe single anyon states (trivial, electric, magnetic, and "combined" state)!
These have a topological property, in the sense that the "direction" of the string is invisible:



So we have

$$\omega_0(F_\xi a F_\xi^*) = \omega_0(B_p F_\xi a F_\xi^* B_p) = \omega_0(F_{\xi'} a F_{\xi'}^*)$$

We may write $\omega_0 \circ \rho_x^k$ where x is the endpoint of the path. Note that the automorphism ρ^k do depend on ξ!

Equivalence of states on $\mathfrak{A}$

Definition

Let ω_1, ω_2 be two pure states. Then we say they are equivalent if the corresponding GNS representations are unitarily equivalent.

Lemma

Two pure states ω_1 and ω_2 on the quasi-local algebra $\mathfrak{A}$ are equivalent if and only if for every $\epsilon > 0$, there is some finite set Λ such that for every local observable A localised outside Λ we have

$$|\omega_1(A) - \omega_2(A)| \leq \epsilon \|A\|.$$

Pure states are inequivalent if they can be distinguished 'at infinity'!

Inequivalence of states

Lemma

The states $\omega_0 \circ \rho_x^k$ and $\omega_0 \circ \rho_y^{k'}$ are inequivalent if $k \neq k'$.

Proof.

We can move the excitations over a finite distance using local unitaries, so wlog we may assume $x = y$. Consider a closed loop ξ. Then one sees that

$$F_\xi = \prod_{p \subset \mathbf{int}(\xi)} B_p,$$

and something similar for closed dual loops.
Since the ribbon operators commute with any A_s and B_p (apart from possibly at the end-point, where they may anti-commute), it follows that $\rho_x^k(F_\xi) = \pm F_\xi$, and $\omega_0 \circ \rho_x^k(F_\xi) = \pm 1$.

Inequivalence of states

Lemma

The states $\omega_0 \circ \rho_x^k$ *and* $\omega_0 \circ \rho_y^{k'}$ *are inequivalent if* $k \neq k'$.

Proof.

(... cont.) The result is -1 only if ξ circles around x, and $k \neq 0$ and ξ is of a different type! Since for any finite set Λ, we can choose a loop surrounding Λ and the endpoint of the semi-infinite ribbon, we can always find an operator $X \in \mathfrak{A}(\Lambda^c)$ with $\|X\| = 1$ such that

$$|\omega_0 \circ \rho_x^k(X) - \omega_0 \circ \rho_x^{k'}(X)| = 2.$$

Since both states are pure, the result follows. $\square$

Some references

[FN15] Leander Fiedler and Pieter Naaijkens, *Haag duality for Kitaev's quantum double model for abelian groups*, Rev. Math. Phys. 27 (2015), no. 9, 1550021, 43.

[Hal06] Hans Halvorson, *Algebraic quantum field theory*, Philosophy of Physics (Jeremy Butterfield and John Earman, eds.), Elsevier, 2006, pp. 731–922.

[Naa17] Pieter Naaijkens, *Quantum spin systems on infinite lattices: a concise introduction*, Lecture Notes in Physics, vol. 933, Springer, Cham, 2017.

[Oga22] Yoshiko Ogata, *A derivation of braided C*-tensor categories from gapped ground states satisfying the approximate Haag duality*, Journal of Mathematical Physics 63 (2022), no. 1, 011902.

Introduction to superselection sector theory II

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The story so far...

We have considered the toric code on the $\mathbb{Z}^2$-lattice and constructed:

- A pure frustration-free ground state ω_0
- Automorphisms ρ^k describing single anyons (electric, magnetic and electromagnetic)
- This gives four equivalence classes of irreducible representations

Questions

- How do we know how to choose which representations?
- Does this set of representations have more structure?

The superselection criterion

Superselection rules

Consider representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ with $\psi_1, \psi_2 \in \mathcal{H}$ unit vectors. Let $\psi_\theta = \frac{\psi_1 + e^{i\theta}\psi_2}{\sqrt{2}}$ and define the state

$$\omega_\theta(A) := \langle \psi_\theta, \pi(A)\psi_\theta \rangle.$$

The (state) vectors ψ_1, ψ_2 satisfy a superselection rule[1] if the expectation values are independent of the relative phase! (This can only happen if π is *not* irreducible!)

Definition

Two states ω_1, ω_2 are called *not superposable* if in any representation π that contains vectors ψ_1, ψ_2 implementing the states, we have that

$$\omega_{\alpha\psi_1+\beta\psi_2}(A) = |\alpha|^2\omega_{\psi_1}(A) + |\beta|^2\omega_{\psi_2}(A) = |\alpha|^2\omega_1(A) + |\beta|^2\omega_2(A)$$

for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

[1] Wick, Wightman and Wigner, Physical Review, 88:101–105, 1952

Superposable irreducible representations

Theorem

Let ω_1, ω_2 be pure states of $\mathfrak{A}$. Then they are superposable iff their GNS representations π_{ω_1} and π_{ω_2} are unitarily equivalent.

- Equivalent representations have the same (normal) states
- Can think of different equivalence classes as describing different 'charges'
- Total charge cannot be changed with (quasi-)local operators!
- The vectors in the representation can describe many excitations (but all have the same 'total charge')

Problem

There are many 'unphysical' irreps of $\mathfrak{A}$!

GNS representations for anyon states

Recall that $(\pi_0, \Omega, \mathcal{H}_0)$ is the GNS representation of the frustration free ground states. Since the maps ρ_x^k are automorphisms of $\mathfrak{A}$, $\pi_0 \circ \rho_x^k$ is again a representation. Moreover, Ω is cyclic for this representation. We have

$$\langle \Omega, \pi_0 \circ \rho_x^k(a) \Omega \rangle = \omega_0(\rho_x^k(a)).$$

Now let ρ_x and ρ'_x be two such automorphisms defined in terms of semi-infinite ribbons ξ and ξ' with the same endpoint. Then $\omega_0 \circ \rho_x = \omega_0 \circ \rho'_x$, so by uniqueness of the GNS representation there must be a unitary $V \in \mathfrak{B}(\mathcal{H}_0)$ such that

$$V \pi_0 \circ \rho_x(a) = \pi_0 \circ \rho'_x(a) V.$$

These are called charge transporters.

The superselection criterion

Definition (Superselection criterion)

Let π_0 be an irreducible “reference” representation of $\mathfrak{A}$. Then π satisfies the superseleciton criterion if

$$\pi \upharpoonright \mathfrak{A}(\Lambda^c) \cong \pi_0 \upharpoonright \mathfrak{A}(\Lambda^c)$$

for all cones Λ.

- Interpretation is that of localisable and transportable representations.
- An equivalence class is called a (superselection) sector
- A general C^*-algebra has many inequivalent representations, but for a given π_0, not many sectors!
- Choice of cone depends on class of models to study.

Sectors of the toric code

Theorem

There are (at least) four irreducible sectors for the toric code.

Proof.

Fix a cone Λ. Choose a semi-infinite path ξ_k for each $k = X, Y, Z$ inside the cone. Then $\pi_0 \circ \rho^k_{\xi_k}(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$. Let Λ' be a different cone, and choose paths $\xi'_k \subset \Lambda'$ as above. Then by independence of the state $\omega_0 \circ \rho^k_{\xi_k}$ on the path (plus moving a charge over a finite distance), it follows that $\pi_0 \circ \rho^k_{\xi_k} \cong \pi_0 \circ \rho^k_{\xi'_k}$. Moreover, from the previous results the four representations $\pi_0 \circ \rho^k_{\xi_k}$ are all inequivalent, and hence in distinct sectors. $\square$

Remark

It turns out these are all irreducible sectors, but we will come back to this later.

What's next

We considered the toric code on the $\mathbb{Z}^2$ lattice:

- Constructed four types of automorphisms ρ_x^k $(k = 0, X, Y, Z)$
- The representations satisfy the superselection criterion:

$$\pi_0 \circ \rho_x^k \upharpoonright \mathfrak{A}(\Lambda^c) \cong \pi_0 \upharpoonright \mathfrak{A}(\Lambda^c)$$

  for all cones Λ
- Representations have the interpretation of describing an anyon
- Anyons are localizable and transportable

We can define extra structure on this set of representations, such as fusion and braiding!

Monoidal/tensor categories

Definition

A *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ together with a distinguished object $1_{\mathcal{C}} \in \mathcal{C}$ and the following families of natural isomorphisms:

1. Associators $\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\simeq} a \otimes (b \otimes c)$
2. Unitors $\lambda_a : 1_{\mathcal{C}} \otimes a \xrightarrow{\simeq} a$ and $\rho_a : a \otimes 1_{\mathcal{C}} \xrightarrow{\simeq} a$

for all $a, b, c \in \mathcal{C}$. These should satisfy the *pentagon* and *triangle* axioms.

Definition

If the associators and unitors are the identity, we say that $\mathcal{C}$ is a strict monoidal category.

Pentagon axiom

$$\begin{tikzcd}
((a \otimes b) \otimes c) \otimes d \arrow[rr, "\alpha_{a\otimes b,c,d}"] \arrow[d, "\alpha_{a,b,c}\otimes \mathrm{id}_d"'] & & (a \otimes b) \otimes (c \otimes d) \arrow[d, "\alpha_{a,b,c\otimes d}"] \\
(a \otimes (b \otimes c)) \otimes d \arrow[dr, "\alpha_{a,b\otimes c,d}"'] & & a \otimes (b \otimes (c \otimes d)) \\
& a \otimes ((b \otimes c) \otimes d) \arrow[ur, "\mathrm{id}_a \otimes \alpha_{b,c,d}"'] &
\end{tikzcd}$$

Triangle axiom

$$
\begin{tikzcd}
(a \otimes 1_{\mathcal{C}}) \otimes b \arrow[rr, "\alpha_{a,1_{\mathcal{C}},c}"'] \arrow[dr, "\rho_a \otimes \mathrm{id}_b"'] & & a \otimes (1_{\mathcal{C}} \otimes b) \arrow[dl, "\mathrm{id}_a \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}
$$

A warm-up

Example

Let $\mathfrak{A}$ be a unital C^*-algebra. Then we can define the category $\mathrm{End}(\mathfrak{A})$ of unital *-endomorphisms of $\mathfrak{A}$, with the following morphisms:

$$\mathbf{Hom}_{\mathrm{End}(\mathfrak{A})}(\rho, \sigma) := \{T \in \mathfrak{A} : T\rho(a) = \sigma(a)T \quad \forall a \in \mathfrak{A}\},$$

with composition the composition of morphisms.
This has a $\otimes$-product, defined objects as $\rho \otimes \sigma := \rho \circ \sigma$. If $S \in \mathbf{Hom}(\rho_1, \rho_2)$ and $T \in \mathbf{Hom}(\sigma_1, \sigma_2)$, define

$$S \otimes T := S\rho_1(T) \in \mathbf{Hom}(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2).$$

This makes $\mathrm{End}(\mathfrak{A})$ into a strict monoidal category. (Exercise!)

Charge transporters

The category $\Delta_{\mathbf{DHR}}$

Motivated by the example $\mathbf{End}(\mathfrak{A})$, we define:

Definition (DHR category, first attempt)

Given a representation π_0, the category $\Delta_{\mathbf{DHR}}$ has as objects endomorphisms ρ of $\mathfrak{A}$ which are

- localised, i.e. there is some cone Λ such that $\rho(a) = a$ for all $a \in \mathfrak{A}(\Lambda^c)$;
- transportable, i.e. for any other cone Λ', there is a ρ' localised in Λ' and a unitary $v \in \mathfrak{B}(\mathcal{H}_0)$ such that

$$v\pi_0(\rho(a)) = \pi_0(\rho'(a))v.$$

The morphisms are the intertwiners, i.e.

$$(\rho, \sigma) := \{s \in \mathfrak{B}(\mathcal{H}_0) : s\pi_0 \circ \rho(a) = \pi_0 \circ \sigma(a)s \quad \forall a \in \mathfrak{A}\}.$$

Note: we have $v \in (\rho, \rho')$ for the charge transporters.

Some remarks on Δ_{DHR}

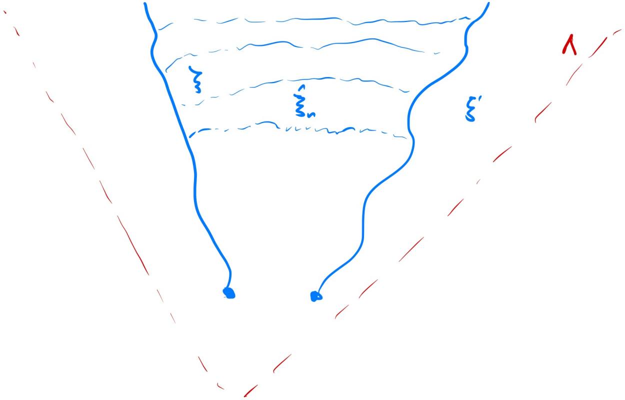
- The automorphisms ρ_x^k defined earlier are in Δ_{DHR}
- Conversely, if $\rho \in \Delta_{\mathrm{DHR}}$, $\pi_0 \circ \rho$ satisfies the superselection criterion
- The category depends on the choice of π_0 via the transportability condition!
- This is not a subcategory of $\mathbf{End}(\mathfrak{A})$ since the morphisms (charge transporters) in Δ_{DHR} need not be in $\pi_0(\mathfrak{A})$
- In particular, monoidal product will be more complicated

Charge transporters

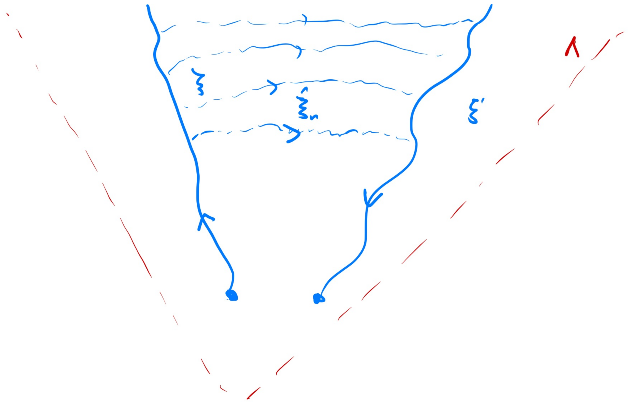
We can explicitly construct charge transporters v:

- Consider semi-finite ribbons $\xi, \xi' \subset \Lambda$ with the same endpoint x and look at the corresponding automorphisms ρ and ρ'
- Write ξ_n (ξ'_n) for the first n edges on the path.
- For each n, choose path $\widehat{\xi}_n$ connecting ends of ξ_n and ξ'_n ...
- ...such that $\mathbf{dist}(\widehat{\xi}_n, x) \to \infty$
- Define $v_n := F_{\xi_n}$. Then $\mathbf{lim}_{n\to\infty} v_n \rho(a) - \rho'(a) v_n = 0$ for all $a \in \mathfrak{A}$.
- Interpretation: v_n moves back the excitation along ξ, then go back along ξ' via $\widehat{\xi}_n$

Charge transporters



Charge transporters



Interlude: von Neumann algebras

Definition

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be a unital $*$-subalgebra. Then $\mathcal{M}$ is called a von Neumann algebra if $\mathcal{M} = \mathcal{M}''$.

Theorem (Bicommutant theorem)

The following are equivalent:

1. *$\mathcal{M}$ is a von Neumann algebra*
2. *$\mathcal{M}$ is closed in the weak operator topology:*
   $x_\lambda \to x$ *wot* $\Leftrightarrow \langle \phi, (x_\lambda - x)\psi \rangle \to 0$ *for all* $\phi, \psi \in \mathcal{H}$.
3. *$\mathcal{M}$ is closed in the strong operator topology:*
   $x_\lambda \to x$ *sot* $\Leftrightarrow \|(x_\lambda - x)\psi\| \to 0$ *for all* $\psi \in \mathcal{H}$.

Charge transporters

Warning

The sequence v_n does not converge (in norm) to an element in $\mathfrak{A}$ in general.

However, $\pi_0(v_n)$ does converge (to a unitary) in the strong operator topology.

Proof (sketch).

It is enough to show that $v_n a\Omega$ is a Cauchy sequence for $a \in \mathfrak{A}_{\mathbf{loc}}$. Note that for n large enough, $\mathbf{supp}\, a \cap \mathbf{supp}(v_n)$ will be constant. For such n, decompose v_n as product of three path operators, such that the middle part has empty intersection with the support of a. Using that $F_\xi \Omega$ only depends on the endpoints of ξ, it follows that for each $a \in \mathfrak{A}_{\mathbf{loc}}$, for $n > k$ with k large enough,

$$v_n a\Omega = F_{\xi_k} F_{\widetilde{\xi}_n} F_{\xi'_k} a\Omega = F_{\xi_k} F_{\xi'_k} a F_{\widetilde{\xi}_n} \Omega \qquad \square$$

Charge transporters

Let Λ be a cone containing the localisation regions of ρ and ρ':

- It follows that $v \in \pi_0(\mathfrak{A}(\Lambda))''$ ...
- ...and in fact $v\pi_0(\rho(a)) = \pi_0(\rho'(a))v$, i.e. $v \in (\rho, \rho')$

Definition

Let Λ be a cone. Then we define the cone von Neumann algebra $\mathcal{R}_\Lambda := \pi_0(\mathfrak{A}(\Lambda))''$.

Haag duality

Definition (Haag duality)

We say a representation π_0 of $\mathfrak{A}$ satisfies Haag duality for cones if $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$. Or in other words, $\mathcal{R}_\Lambda = \mathcal{R}'_{\Lambda^c}$.

Theorem (Fiedler-PN)

Haag duality for cones holds in all abelian quantum double models.

Remark: the direction $\pi_0(\mathfrak{A}(\Lambda))'' \subset \pi_0(\mathfrak{A}(\Lambda^c))'$ always holds by locality.

Remark

In the example of the toric code, we can construct everything explicitly and Haag duality is only necessary to show completeness.

Application I: localisation of intertwiners

Lemma

Let Λ_1 and Λ_2 be two cones both contained in a larger cone Λ, and suppose that ρ_i is localised in Λ_i. That is, $\rho_i(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda_i^c)$. If $v \in (\rho_1, \rho_2)$, then $v \in \pi_0(\mathfrak{A}(\Lambda))''$.

Proof.

Consider $a \in \mathfrak{A}(\Lambda^c)$. Then we have

$$v\pi_0(a) = v\rho_1(a) = \rho_2(a)v = \pi_0(a)v,$$

where we used that the ρ_i are localised in Λ twice. But this implies $v \in \pi_0(\mathfrak{A}(\Lambda^c))' = \pi_0(\mathfrak{A}(\Lambda))''$ by Haag duality. $\square$

Application II: localised repns

Lemma

Suppose that π *satisfies the superselection criterion. Then for any cone* Λ*, there is an equivalent representation* $\rho_\Lambda : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ *such that* $\rho_\Lambda(a) = \pi_0(a)$ *for all* $a \in \mathfrak{A}(\Lambda^c)$*. Moreover, if* $a \in \mathfrak{A}(\Lambda)$*, then* $\rho_\Lambda(a) \in \pi_0(\mathfrak{A}(\Lambda))''$*.*

Proof.

By the superselection criterion, there is a unitary $v : \mathcal{H} \to \mathcal{H}_0$ such that $v\pi(a)v^* = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$. Define $\rho_\Lambda(a) = v\pi_0(a)v^*$. Then $\rho_\Lambda : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ is a representation. Moreover, if $a \in \mathfrak{A}(\Lambda)$ and $b \in \mathfrak{A}(\Lambda^c)$, we have

$$\pi_0(b)\rho_\Lambda(a) = v\pi(b)v^*v\pi(a)v^* = v\pi(ba)v^* = v\pi(ab)v^* = \rho_\Lambda(a)\pi_0(b).$$

The claim then follows by Haag duality. □

Localised representations

By construction, for the "anyon automorphisms" we defined, $\pi_0 \circ \rho(\mathfrak{A}(\Lambda)) \subset \pi_0(\mathfrak{A}(\Lambda))$, and in fact $\rho : \mathfrak{A} \to \mathfrak{A}$.

For an arbitrary representation π satisfying the superselection criterion we can get a unitary equivalent representation $\rho : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ such that $\rho(a) = \pi_0(a)$ for all $a \in \mathfrak{A}(\Lambda^c)$, where Λ is the localisation region of ρ. But in general $\rho(\mathfrak{A}) \subset \pi_0(\mathfrak{A})$ is not true, i.e. we cannot restrict to endomorphisms of $\mathfrak{A}$.

However, we still get good control over the localisation, namely $\rho(\mathfrak{A}(\Lambda)) \subset \pi_0(\mathfrak{A}(\Lambda))''$.

Introduction to superselection sector theory III

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QFT and Topological Phases via Homotopy Theory and Operator Algebras

30 June – 3 July, 2025

The monoidal product

Monoidal structure

Recall that for $\mathbf{End}(\mathfrak{A})$, $s \in (\rho_1, \rho_2)$ and $t \in (\sigma_1, \sigma_2)$:

$$\rho \otimes \sigma = \rho \circ \sigma, \qquad s \otimes t = s\rho_1(t).$$

However we cannot define the monoidal product on Δ_{DHR} as for $\mathbf{End}(\mathfrak{A})$:

- The intertwiners (charge transporters) of interest are in $\pi_0(\mathfrak{A}(\Lambda))''$ for some sufficiently large cone. But in general this is not in $\pi_0(\mathfrak{A})$! So $\rho(v)$ need not be defined.
- More generally, a superselection sector does not lead to an endomorphism of $\mathfrak{A}$, as ρ_Λ maps $\mathfrak{A}(\Lambda)$ into a bigger algebra. (although the examples we constructed all have this property).

Auxiliary algebra

We need to make sense of $\rho(v)$ where $v \in \mathfrak{A}(\Lambda)''$: extend ρ to a bigger algebra that contains the intertwiners.
Fix a "forbidden direction" by choosing a cone an angle $\theta \in [0, 2\pi)$ and $0 < \phi < \pi$.[1] Let $\mathcal{C}(\theta, \phi)$ be the set of cones whose 'angles' don't intersect $(\theta - \phi, \theta + \phi) \pmod{2\pi}$. That is, all cones that 'point in a different direction'.

The set $\mathcal{C}(\theta, \phi)$ is directed and $\bigcup_{\Lambda \in \mathcal{C}(\theta,\phi)} \Lambda = \mathbb{R}^2$.

Then define

$$\mathfrak{A}^{\mathrm{aux}} := \overline{\bigcup_{\Lambda \in \mathcal{C}(\theta,\phi)} \mathfrak{A}(\Lambda)''}^{\|\cdot\|}$$

Note that $\mathfrak{A} \subset \mathfrak{A}^{\mathrm{aux}}$.

[1] Following Ogata, *J. Math. Phys.* 63, 2022

Extensions to the auxiliary algebra

Lemma

Let $\rho : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ be localised and transportable. Then ρ can be extended to a $*$-homomorphism $\rho^a : \mathfrak{A}^{\mathbf{aux}} \to \mathfrak{A}^{\mathbf{aux}}$. This extension is weak-operator continuous on the algebras $\mathfrak{A}(\Lambda)''$.

Proof (sketch).

Let Λ be a cone in $\mathcal{C}(\theta, \phi)$. Let Λ' be a cone disjoint from Λ. By localisability, there is ρ' localised in Λ' and a unitary v such that $\rho(a) = v\rho'(a)v^*$. But then if $a \in \mathfrak{A}(\Lambda)$, we have

$$\rho(a) = v\rho'(a)v^* = v\pi_0(a)v^*.$$

But conjugation is weak operator continuous, so can extend to the weak closure $\mathfrak{A}(\Lambda)''$. The image is seen to be in $\mathfrak{A}^{\mathbf{aux}}$ using Haag duality. $\square$

Auxiliary algebra

- Category obtained in following can be shown to be independent of choice of forbidden direction
- Technique is similar to ‘puncturing the circle’
- Various other approaches:
  - Universal algebra (Fredenhagen)
  - ‘Coordinate charts’ (Fröhlich-Gabbiani)
  - Prefactorisation algebras (see talk Schenkel next week)
  - Nets of representations (Gabbiani-Fröhlich, Bhardwaj et al.)

The category $\Delta_{\mathbf{DHR}}$

Definition (DHR category)

Given a representation π_0, the category Δ_{DHR} has as objects $*$-homomorphisms $\rho : \mathfrak{A} \to \mathfrak{A}^{\mathrm{aux}}$ which are

- localised, i.e. there is some cone Λ such that $\rho(a) = a$ $(= \pi_0(a))$ for all $a \in \mathfrak{A}(\Lambda^c)$;
- transportable, i.e. for any other cone Λ', there is a ρ' localised in Λ' and a unitary $v \in \mathfrak{B}(\mathcal{H}_0)$ such that

$$v\rho(a) = \rho'(a)v.$$

The morphisms are the intertwiners, i.e.

$$(\rho, \sigma) := \{s \in \mathfrak{B}(\mathcal{H}_0) : s\rho(a) = \sigma(a)s \quad \forall a \in \mathfrak{A}\}.$$

Δ_{DHR} depends on the choice of π_0!

The tensor product

We can now define a tensor product ($s \in (\rho_1, \rho_2)$, $t \in (\sigma_1, \sigma_2)$):

$$\rho \otimes \sigma := \rho^a \circ \sigma, \qquad s \otimes t := s\rho_1^a(t).$$

This is now well-defined! Moreover, the result is localised and transportable again.

Proposition

The category Δ_{DHR} is a tensor/monoidal category.

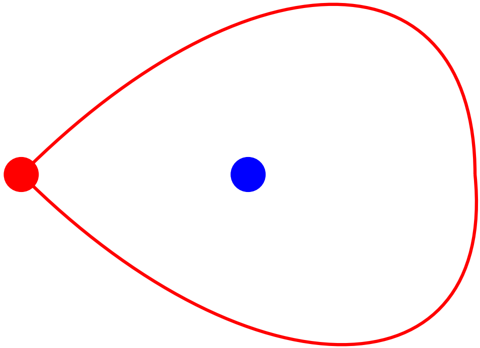
Proof.

Now everything is well-defined using the auxiliary algebra, this is a straightforward calculation. $\square$

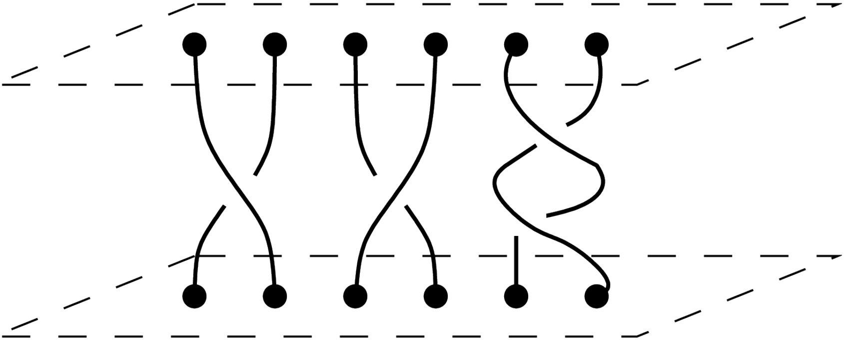
- The tensor unit is $\iota := \mathbf{id}_{\mathfrak{A}}$, i.e. the vacuum sector.
- Clear physical interpretation

Braiding

Anyon interchange



Anyon interchange



Braiding

Definition

Let $\mathcal{C}$ be a monoidal category. A *braiding* on $\mathcal{C}$ is a family of natural isomorphisms

$$\varepsilon_{a,b} \in \mathbf{Hom}_{\mathcal{C}}(a \otimes b, b \otimes a)$$

such that following diagrams commute (next slides).

- $\varepsilon_{a,b}$ is called a *symmetry* if $\varepsilon_{a,b} \circ \varepsilon_{b,a} = \mathbf{id}_{b \otimes a}$
- Can define a braided functor in the obvious way

Example

- The category of vector spaces with linear maps, with the braiding given by the tensor flip (this is a symmetry)
- The representation category of a quasi-triangular Hopf-algebra

Hexagon axioms

$$
\begin{tikzcd}
 & a \otimes (b \otimes c) \arrow[r, "\varepsilon_{a,b\otimes c}"] & (b \otimes c) \otimes a \arrow[dr, "\alpha_{b,c,a}"] & \\
(a \otimes b) \otimes c \arrow[ur, "\alpha_{a,b,c}"] \arrow[dr, "\varepsilon_{a,b}\otimes \mathrm{id}_c"'] & & & b \otimes (c \otimes a) \\
 & (b \otimes a) \otimes c \arrow[r, "\alpha_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[ur, "\mathrm{id}_b \otimes \varepsilon_{a,c}"'] &
\end{tikzcd}
$$

The same should hold for ε replaced by ε^{-1}

Braiding

Consider again $\mathbf{End}_{\mathcal{A}}$. In general, for two endomorphisms ρ and σ, there needs to be no relation between $\rho \circ \sigma$ and $\sigma \circ \rho$. So no braiding can be defined!

Because objects in Δ_{DHR} are localised and transportable, we can define a braiding on Δ_{DHR}!

Remark

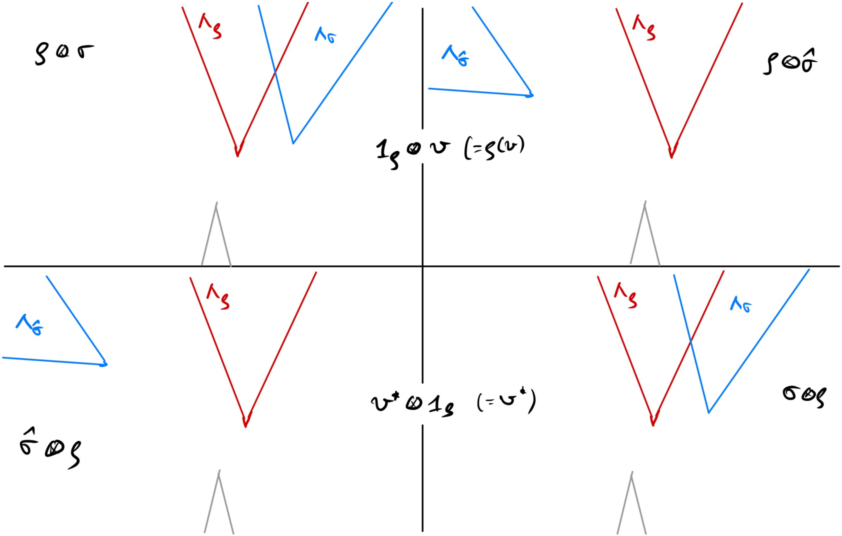
As will become clear from the construction, it works because of the geometric properties of the cones. This geometric origin can be made much more explicit (see e.g. talk Schenkel next week).

Braiding: construction

- Consider ρ, σ localised in Λ_ρ and Λ_σ respectively.
- Choose a cone $\widehat{\Lambda}_\sigma$ to the left of Λ_ρ (this can be defined unambiguously using the 'forbidden direction')
- There exists $\widehat{\sigma}$ localised in $\widehat{\Lambda}_\sigma$ and a unitary $v \in (\sigma, \widehat{\sigma})$
- By localisation, $\rho \otimes \widehat{\sigma} = \widehat{\sigma} \otimes \rho$ since they act non-trivially on disjoint parts of the system!
- Hence $\varepsilon_{\rho,\sigma} := (v^* \otimes \mathbf{id})(\mathbf{id}_\rho \otimes v) = v^* \rho(v) \in (\rho \otimes \sigma, \sigma \otimes \rho)$

The isomorphims are independent of the choices made and define a braiding on Δ_{DHR}.

Braiding

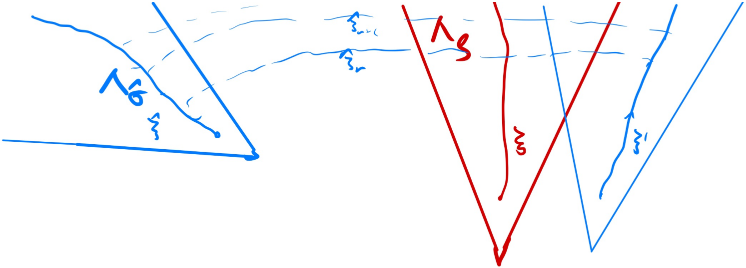


Braiding in the toric code

The explicit construction of charge transporters and automorphisms in the toric code allow for explicit calculation of the braiding operators!
Main step is to calculate $\rho^a(v)$, where v transports σ:

- $v = \mathbf{lim}\, v_n$ in the weak operator topology
- ρ^a is weak operator continuous: $\rho^a(v) = \mathbf{lim}_n\, \rho(v_n)$
- But v_n is a local observable
- Can compute $\rho(v_n)$ explicitly and obtain $\rho^a(v) = \pm v$

Braiding in the toric code



$$g(v) = \lim_n g(F_{\xi_n})$$

$$= \lim_n \lim_k F_\xi F_{\xi_n}^{\ast} F_\xi^{*}$$

Braiding in the toric code

The braiding is given as follows on the four sectors (sign depends on the 'relative position'):

$\varepsilon_{\rho_1,\rho_2}$	X	Y	Z
X	I	$\pm I$	$\pm I$
Y	$\mp I$	$-I$	$\pm I$
Z	$\mp I$	$\mp I$	I

In particular $\varepsilon_{X,Z} \circ \varepsilon_{Z,X} = -I$ (abelian anyons!)

Fusion

Fusion categories

Consider $\mathcal{C} = \mathrm{Rep}_f G$, where G is a finite group. This is a semi-simple monoidal category:

$$\pi_i \otimes \pi_j \cong \bigoplus_{\pi_k \in \mathrm{Irr}(\mathcal{C})} N_{ij}^k \pi_k$$

For the toric code, we have something similar:

$$\rho_X \otimes \rho_X \cong \rho_Z \otimes \rho_Z \cong \iota \qquad \rho_X \otimes \rho_Z \cong \rho_Y \qquad \text{etc...}$$

In this case it is easy because the anyons are abelian. But we can do something more generally!

Linear structure

Consider the category Δ_{DHR}. We have a ('unitary') $*$-structure:

- All Hom-sets in the category are vector spaces over $\mathbb{C}$
- The adjoint of $\mathfrak{B}(\mathcal{H}_0)$ gives an anti-linear contravariant functor $* : \Delta_{\mathrm{DHR}} \to \Delta_{\mathrm{DHR}}$ such that $*(\rho) = \rho$ for all objects and $(\rho, \sigma) \ni v \mapsto v^* \in (\sigma, \rho)$.
- In fact (ρ, ρ) is a C*-algebra with norm inherited from $\mathfrak{B}(\mathcal{H}_0)$

This means we can talk about (partial) isometries, orthogonal projections and unitaries in the category. We can attempt to define direct sums and for $p \in (\rho, \rho)$ an orthogonal projection, subobjects.

Technical remark

Essentially (up to minor technical details), we want to show that Δ_{DHR} is an additive category.

Interlude: factors

Let $\mathcal{M}$ be a factor: a vNA with $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$.

Definition

A trace on $\mathcal{M}$ is a linear map $\tau : \mathcal{M}_+ \to [0, \infty]$ such that $\tau(aa^*) = \tau(a^*a)$ and $\tau(\lambda x) = \lambda\tau(x)$ for $\lambda \geq 0$.

On each factor there exists a trace that is non-zero on Factors can be classified according to which values a trace can take on projections:

- Type I_n: $\{0, 1, \ldots, n\}$ with $n \in \mathbb{N} \cup \{\infty\}$. In this case, $\mathcal{M} \simeq \mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.
- Type II_1: $[0, 1]$.
- Type II_∞: $[0, \infty]$.
- Type III: $\{0, \infty\}$.

A factor is called properly infinite if ∞ is in the range of the trace.

Direct sums

If ρ_1, ρ_2 are objects in Δ_{DHR}, and v_i are isometries such that $v_1 v_1^* + v_2 v_2^* = I$, then

$$(\rho_1 \oplus \rho_2)(a) := v_1 \rho_1(a) v_1^* + v_2 \rho_2(a) v_2^*$$

is again a representation of $\mathfrak{A}$.

Lemma (PN, Ogata)

For the toric code $\pi_0(\mathfrak{A}(\Lambda))''$ is a Type II$_\infty$ factor. In particular, there exist $v_i \in \pi_0(\mathfrak{A}(\Lambda))''$ with $v_1 v_1^* + v_2 v_2^* = I$ and $v_i^* v_i = I$.

Corollary

The category Δ_{DHR} has direct sums.

Note: it follows that $v_i \in (\rho_i, \rho_1 \oplus \rho_2)$

Subobjects

Consider some $\rho \in \Delta_{\mathbf{DHR}}$ and $p \in (\rho, \rho)$ an orthogonal projection. A subobject for this pair is an object $\sigma \in \Delta_{\mathbf{DHR}}$ together with a $v \in (\sigma, \rho)$ such that $v^* v = I$ and $vv^* = p$.

This is the analogue of restricting a representation to an invariant subspace.

Lemma (Ogata, arXiv:2106.15741)

If $p \in (\rho, \rho)$ and ρ is localised in Λ, there is some isometry $v \in \mathfrak{A}(\Lambda)''$ with $vv^* = p$.

Corollary

The category Δ_{DHR} has subobjects.

Proof.

Choose $\sigma(a) = v^* \rho(a) v$. $\square$

Fusion rules

Note that ρ_i are subobjects of $\rho_1 \oplus \rho_2$.
If $p \in (\rho, \rho)$ is a non-trivial projection, can get subobjects ρ_1 (ρ_2) corresponding to p (resp. $(1-p)$), and $\rho = \rho_1 \oplus \rho_2$.

Definition

An object in Δ_{DHR} is called irreducible or simple if $(\rho, \rho) = \mathbb{C}I$.

Want to obtain fusion rules: for ρ_i, ρ_j irreducible:

$$\rho_i \otimes \rho_j = \bigoplus_{k \in \mathbf{Irr}(\Delta_{\mathrm{DHR}})} N_{ij}^k \rho_k$$

In general, it is not guaranteed that there are only finitely many sectors, or that there is a decomposition into finitely many irreducibles (i.e., the category is semi-simple).

Fusion rules in the toric code

The four sectors we have found have very simple fusion rules, e.g.

$$\rho^X \otimes \rho^X \cong \mathbf{id} \cong \rho^Z \otimes \rho^Z \qquad \rho^X \otimes \rho^Z \cong \rho^Y.$$

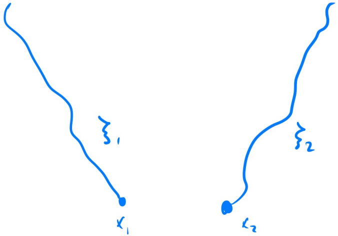
Note that all these should be understood up to unitary equivalence.

For example, take two semi-infinite "X-strings" ξ_1 and ξ_2 with different endpoints. Then

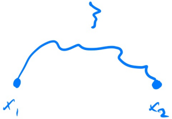
$$\rho_{\xi_1} \otimes \rho_{\xi_2} \cong \mathrm{Ad}(F_\xi) \cong \mathbf{id}$$

Physically, $\omega(\rho_{\xi_1} \otimes \rho_{\xi_2}(a))$ describes a state with two X-excitations. But this has trivial total charge, and hence is in the trivial sector.

Fusion of two X-anyons



$$\omega(\rho_{\xi_1} \otimes \rho_{\xi_2}(a)) = \omega(F_\xi \, a \, F_\xi^*)$$



This is in the trivial sector!

Some remarks

- The existence of direct sums and subobjects follows from quite general assumptions. In particular, they hold for pure ground states of bounded finite range interactions (on $\mathbb{Z}^2$) satisfying a spectral gap condition.[2]
- We can construct models where the cone algebras are Type II_∞ or Type III (and even provide the finer III_λ classification). Models with the same superselection sectors can have different type of cone algebras![3]
- If $\mathbf{dim}(\rho, \rho) < \infty$, it follows that ρ can be decomposed into irreducibles.

[2] Ogata, *J. Math. Phys.* 63 (2023)

[3] Jones, PN, Penneys, Wallick (+ appendix Izumi). arXiv:2307.12552. To appear in FoM Sigma.

Introduction to superselection sector theory IV

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QFT and Topological Phases via Homotopy Theory and Operator Algebras

30 June – 3 July, 2025

Further structure

Duality / rigidity

Definition

A conjugate for $\rho \in \Delta_{\mathbf{DHR}}$ is a triple $(\overline{\rho}, r, \overline{r})$ with $r \in (\mathbf{id}, \overline{\rho} \otimes \rho)$ and $\overline{r} \in (\mathbf{id}, \rho \otimes \overline{\rho})$ such that

$$(\overline{r}^* \otimes \mathbf{id}_\rho)(\mathbf{id}_\rho \otimes r) = \mathbf{id}_\rho \qquad (r^* \otimes \mathbf{id}_{\overline{\rho}})(\mathbf{id}_{\overline{\rho}} \otimes \overline{r}) = \mathbf{id}_{\overline{\rho}}\,.$$

This gives a conjugate charge, and allows us to define the quantum dimension.

Warning

In the toric code (and many other models), we can explicitly construct conjugates. However, it is not known if conjugates automatically exist. In QFT, there are examples of sectors for which a dual does not exist.

Completeness

Have we found all sectors?

We have constructed four different (irreducible) sectors for the toric code (as expected)!

Question

Are these all irreducible sectors? That is, are there irreducible representations satisfying the superselection criterion that are not equivalent to one of these four?

Theorem (Bols-Vadnerkar, arXiv:2310.19661)

For each irreducible representation (C, ρ) *of* $D(G)$ *there is an anyon sector* $\pi^{(C,\rho)}$*. The representations* $\{\pi^{(C,\rho)}\}_{(C,\rho)}$ *are pairwise disjoint, and any anyon sector is unitarily equivalent to one of them.*

The category $\Delta_{\mathbf{DHR}}$

The quantum double $D(G)$ of a finite group G is a certain Hopf algebra constructed from $\mathbb{C}[G]$:

- $\mathbf{Rep}_f\, D(G)$ is a modular tensor category.
- Irreps are in 1-1 correspondence with pairs (C, ρ), where C is a conjugacy class of G, and ρ an irreducible representation of the centraliser of some $g \in C$.
- In particular, for $G = \mathbb{Z}_2$ there are four irreps.

Theorem

The category Δ^f_{DHR} (i.e., restrict to finite direct sums) for the toric code is braided tensor equivalent to $\mathbf{Rep}\, D(\mathbb{Z}_2)$.

Proof sketch.

We have representatives of the four sectors, which we can map to irreps of $D(\mathbb{Z}_2)$. These are all irreducible sectors. $\square$

Non-abelian quantum double model

The quantum double model can be defined for any finite group G (with local Hilbert space $\mathcal{H}_x = \mathbb{C}^{|G|}$).

- If G abelian, analysis is very similar to toric code
- For non-abelian, can define ribbon operators similar to path operators
- However, if irrep of $D(G)$ has $d = \mathbf{dim} > 1$, corresponding ribbon operators come in multiplets
- Easier to construct amplimorphisms $\rho : \mathfrak{A} \to M_d(\mathfrak{A})$
- Can be mapped back to endomorphisms of $\mathfrak{A}^{\mathbf{aux}}$ using properly infiniteness of cone algebra
- Obtain the category $\mathbf{Rep}_f\, D(G)$ as expected.[1]

[1] Bols, Hamdan, PN, Vadnerkar, arXiv:2503.15611

Open questions

- Is this category indeed an invariant?
- When can we prove conjugates exist? (Note: if ρ has a conjugate, it follows that $\mathbf{dim}(\rho, \rho) < \infty$)
- Can we find (physically meaningful) conditions that guarantee that the category is modular?
- Which states lead to an interesting superselection sector theory?

Stability

Approximately factorisable autom.

Definition (Informal)

Let $\alpha \in \mathbf{Aut}(\mathfrak{A})$ be an automorphism. We say it is approximately factorisable[a] if for any cone Λ and $\delta > 0$

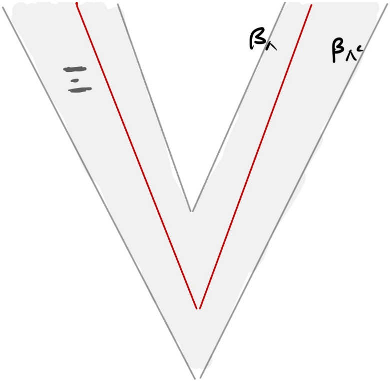
1. There are automorphims $\beta_\Lambda, \widetilde{\beta}_\Lambda$ of $\mathfrak{A}(\Lambda)$ and similarly for Λ^c, together with automorphisms $\Xi, \widetilde{\Xi}$ acting only 'near the boundary' of Λ and unitaries $v, \widehat{v}$ such that:

$$\alpha = \mathbf{Ad}(v) \circ \Xi \circ (\beta_\Lambda \otimes \beta_{\Lambda^c})$$
$$\alpha^{-1} = \mathbf{Ad}(\widetilde{v}) \circ \widetilde{\Xi} \circ (\widetilde{\beta}_\Lambda \otimes \widetilde{\beta}_{\Lambda^c})$$

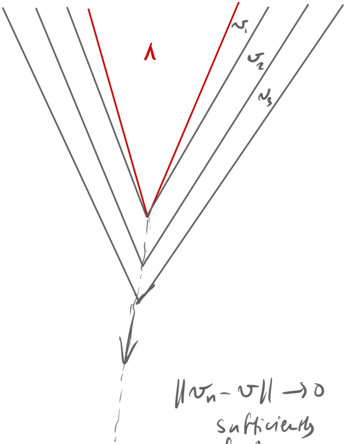
2. The unitaries v and $\widetilde{v}$ can be approximated by unitaries in translates of cones slightly wider opening angle than Λ_δ with error decaying fast enough.

[a] See Ogata, arXiv:2106.15741 for details

Approximately factorisable autom.



$$\alpha = \mathrm{Ad}(v)\circ \Xi \circ (\beta_\Lambda \otimes \beta_{\Lambda^c})$$



$\|v_n - v\| \to 0$
sufficiently fast

Where do these come from?

- These are a generalisation of finite-depth quantum circuits
- Also come from quasi-local automorphisms, i.e. $\alpha \in \mathbf{Aut}(\mathfrak{A})$ satisfying a suitable Lieb-Robinson bound.

Theorem

Let H_Λ be some gapped local dynamics satisfying the local topological order conditions. Let $H'_\Lambda = H_\Lambda + \Phi_\Lambda$, where Φ_Λ is some sufficiently small perturbation. Then H'_Λ is gapped as well, and the ground states (obtained as weak-$*$ limits of local ground states) of both models are related by a approximately factorisable automorphism.

This is a combination of results by Bravyi, Hastings, Michalakis & Zwolak (JMP 51, 2010 and CMP 322, 2013) and Bachmann, Nachtergaele, Michalakis and Sims (CMP 309, 2012).

Phases of matter

Definition

Two states ω_1, ω_2 are said to be in the same (quantum) phase if there is some approximately factorisable automorphism α such that $\omega_1 \circ \alpha = \omega_2$.

Question

Suppose that ω_1 and ω_2 are in the same phase. How are the categories $\Delta_{\mathbf{DHR}}(\omega_1)$ and $\Delta_{\mathbf{DHR}}(\omega_2)$ related?

Approximate Haag duality

Because α in general is not an automorphism of cone algebras, $\pi_0 \circ \alpha$ needs not satisfy Haag duality for cones even if π_0 does.
$\Rightarrow$ need a weaker notion!

Definition (Approximate Haag duality)

A representation π_0 of $\mathfrak{A}$ is said to satisfy approximate Haag duality (for cones) if for every cone Λ and $\epsilon > 0$ small enough, there is a unitary U_{Λ_ϵ} (in $\mathfrak{B}(\mathcal{H}_0)$) and $R_\epsilon > 0$ such that

$$\pi_0(\mathfrak{A}(\Lambda^c))' \subset U_{\Lambda,\epsilon}\pi_0(\mathfrak{A}((\Lambda - R_\epsilon)_\epsilon))'' U_{\Lambda,\epsilon}^*.$$

Moreover, $U_{\Lambda,\epsilon}$ can be approximated by unitaries in the cone von Neumann algebras with error going to zero fast enough.

Approximate Haag duality

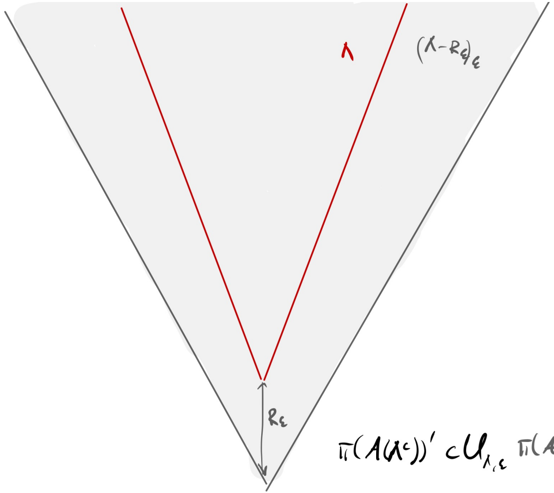
Remark

If π satisfies Haag duality for cones, it also satisfies approximate Haag duality.

Proposition (Ogata)

Approximate Haag duality is stable, in the sense that if π satisfies approximate Haag duality, then so does $\pi \circ \alpha$.

Approximate Haag duality



$$\pi(\mathcal{A}(\Lambda^c))' \subset U_{\Lambda,\varepsilon}\, \pi(\mathcal{A}((\Lambda - R_\varepsilon)_\varepsilon))'' \, U_{\Lambda,\varepsilon}^*$$

Sector theory with approximate Haag duality

Approximate Haag duality is enough to develop a sector theory along the same lines as before.

- Superselection criterion is the same
- $\mathfrak{A}^{\text{aux}} := \overline{\bigcup_{\Lambda \in \mathcal{C}(\theta,\phi)} \pi_0(\mathfrak{A}(\Lambda^c))'}^{\|\cdot\|}$
- Don't get strict localisation of representations, but get decaying tails
- But image still ends up in $\mathfrak{A}^{\text{aux}}$!
- Same is true for intertwiners
- Braiding is more complicated, as $\rho \circ \sigma \neq \sigma \circ \rho$ if localised in distinct cones due to decaying tails: have to do a limiting procedure

Stability

Theorem (Ogata, arXiv:2106.15741)

Let Φ_1, Φ_2 be two uniformly bounded finite range interactions on $\mathbb{Z}^2$ with pure gapped ground states ω_i. Suppose that there is an approximate factorisable automorphism α such that $\omega_1 = \omega_2 \circ \alpha$ and that one (and hence both) of the GNS representations π_{ω_i} satisfies approximate Haag duality. Then the corresponding DHR catgories are unitarily braided monoidally equivalent.

Hence the category of superselection sectors is an invariant of the phase![2]

[2] See also Cha, PN, Nachtergaele, arXiv:1804.03203

Long-range entanglement

Product states

The existence of anyons in the toric code is possible because the state has long-range entanglement.

Definition

Let ω be a pure state on $\mathfrak{A}$. Then ω is a product state with respect to a cone Λ if there are states ω_Λ on $\mathfrak{A}_\Lambda$ and ω_{Λ^c} on $\mathfrak{A}(\Lambda^c)$ such that $\omega \approx \omega_\Lambda \otimes \omega_{\Lambda^c}$.

Proposition

Let ω be a pure product state with respect to some cone Λ. Then $\pi_\omega(\mathfrak{A}(\Lambda))''$ is a Type I factor and we have $\mathcal{R}_\Lambda = \mathcal{R}'_{\Lambda^c}$.

Proof sketch.

The GNS representation factorises as $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^c}$, with $\pi_\omega(\mathfrak{A}(\Lambda))$ acting on the first factor, and $\pi_\omega(\mathfrak{A}(\Lambda^c))$ on the second. By irreducibility of π_ω, $\pi_\omega(\mathfrak{A}(\Lambda))'' = \mathfrak{B}(\mathcal{H}_\Lambda) \otimes I$. We see Haag duality by taking commutants. $\square$

Sector theory

Theorem (PN, Ogata arXiv:2102.07707)

Let ω be a pure state which is a product state with respect to some cone Λ. Then the sector theory with respect to π_ω is trivial, in the sense that every representation satisfying the SSC is a (possibly infinite) direct sum of copies of π_ω.

Proof sketch.

Localise π in Λ. Then we get an equivalent π_Λ and a normal endomorphism $\pi_\Lambda : \mathcal{R}_\Lambda \to \mathcal{R}_\Lambda$. But this must be some direct sum of the identity representation. $\square$

- We only need ω being product with respect to a single cone, however the superselection criterion must hold for all cones.
- It is important that ω is pure, which implies $\mathcal{R}_\Lambda$ is a factor.

Long-range entanglement

Definition

A state ω is said to be long-range entangled if $\omega \circ \alpha$ is not a product state with respect to a cone for any quasi-factorisable automorphism α.

Corollary

If ω is not long-range entangled, the corresponding superselection structure $\Delta_{DHR}(\omega)$ is trivial.

Thus you need long-range entanglement to get anyons!

Completeness: alternative approach

Completeness: alternative approach

The previous result relies on detailed knowledge of the quantum double model and its frustration-free ground state.

There is an alternative approach based on the Jones index of a certain subfactor.

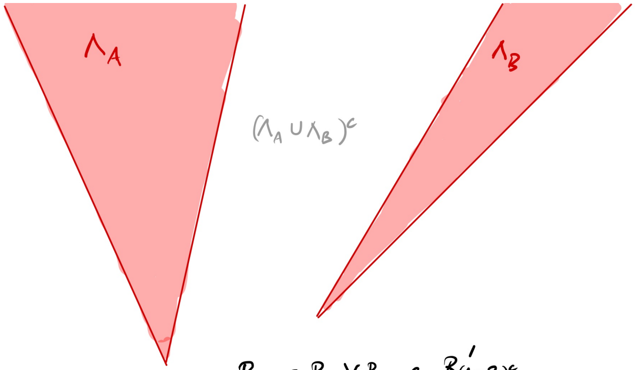
This is based on analogous results for rational conformal field theory.[3]

- Consider two disjoint cones Λ_A and Λ_B such that...
- .. there is a Type I factor $\mathcal{N}$ such that $\mathcal{R}_{\Lambda_A} \subset \mathcal{N} \subset \mathcal{R}'_{\Lambda_B}$.
- This holds for quantum double models
- Can define two von Neumann algebras:

$$\mathcal{R}_{AB} := \mathcal{R}_{\Lambda_A} \vee \mathcal{R}_{\Lambda_B}, \qquad \widehat{\mathcal{R}}_{AB} := \mathcal{R}'_{(\Lambda_A \cup \Lambda_B)^c}$$

[3] Kawahigashi, Longo, Müger, *Commun. Math. Phys.* 219 (2001)

Cone algebras



$$R_{AB} = R_A \vee R_B \subseteq R'_{(A \cup B)^c}$$

Relation between the two algebras

- From locality it follows that $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$
- However in general the two are not equal (even if Haag duality holds)!
- From irreducibility of π_0, it follows that $\mathcal{R}_{AB}$ and $\widehat{\mathcal{R}}_{AB}$ are factors
- … and ${\mathcal{R}'}_{AB} \cap \widehat{\mathcal{R}}_{AB} = \mathbb{C}I$.
- Hence $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$ is an irreducible subfactor
- Let $\rho_{A,B}$ be localised in $\Lambda_{A,B}$ and $v \in (\rho_A, \rho_B)$ unitary. Then $v \in \widehat{\mathcal{R}}_{AB}$
- But in general $v \notin \mathcal{R}_{AB}$!

Relative sizes

Main idea

The algebra $\widehat{\mathcal{R}}_{AB}$ is bigger because it contains the charge transporters. Hence if we can "quantify" how much bigger, we may learn something about the number of sectors.

We can use the Jones index $[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}]$:

- $[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}] \geq 1$ with equality iff $\widehat{\mathcal{R}}_{AB} = \mathcal{R}_{AB}$
- If $[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}] < \infty$, there are $b_0, \ldots, b_n \in \widehat{\mathcal{R}}_{AB}$ such that

$$\widehat{\mathcal{R}}_{AB} = \left\{ \sum_{i=0}^{n} a_i b_i : a_i \in \mathcal{R}_{AB} \right\}.$$

Upper bound on number of sectors

Theorem (PN, J Math Phys 54 (2013))

The number of (irreducible) superselection sectors is bounded from above by

$$\mu_{AB} := \inf_{\Lambda_A \cup \Lambda_B} [\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}].$$

If each sector has a conjugate, we have $\sum_i d(\rho_i)^2 \leq \mu_{AB}$.

- For abelian quantum double models, we can show $\widehat{\mathcal{R}}_{AB} \rtimes_\alpha (G \times G)$.
- It follows $[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}] = 4$ for the toric code.
- Since we have constructed four sectors, these must be all!

Summary

Summary

- Can obtain a braided C*-category of superselection sectors from first principles from local dynamics
- … using only a few assumptions ((approximate) Haag duality and ground state gap)
- Anyons can be identified with representations satisfying $\pi \upharpoonright \mathfrak{A}(\Lambda^c) \cong \pi_0 \upharpoonright \mathfrak{A}(\Lambda^c)$
- Category can be constructed explicitly in models such as the toric code or quantum double, and gives the expected result
- The category is an invariant of the quantum phase
- Need long-range entanglement to get non-trivial sectors!