Local topological order and boundary algebras

Pieter Naaijkens

Cardiff University

Joint with Corey Jones, David Penneys, and Daniel Wallick

arXiv:2307.12552 & arXiv:2506.19969

DPG Frühjahrstagung, 17 March 2026

Motivation

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Main issue: the classification of gapped ground states:

- We will always work in the thermodynamic limit
- Gapped ground states of local Hamiltonians...
- ... with some equivalence relation
- Focus on states with topological order (or long-range entanglement)

Question

Can we find (physically interesting) invariants?

Why are these states interesting?

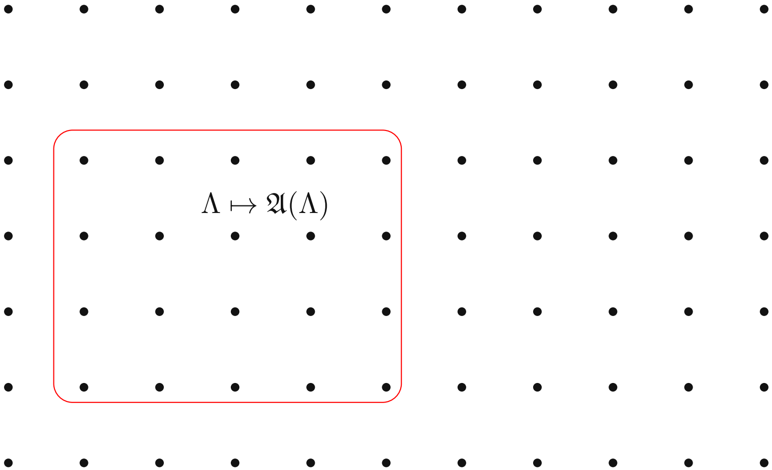
- Can host anyons: quasi-particles/superselection sectors/charges/... with braided statistics
- Algebraic properties of anyons are described by braided tensor C*-categories (typically even modular or braided fusion)
- 'Topological' nature makes these properties robust
- In other words, the category should be an invariant

Question

What can we learn about the bulk anyon theory from looking at the boundary?

Setup and main result

Quantum lattice systems



Quantum spin systems

Let S be a countable discrete set. A *quantum spin system* on S is given by a unital $\mathbf{C}^*$-algebra $\mathfrak{A}$ and:

- for $\Lambda \subset S$ finite, a subalgebra $\mathfrak{A}(\Lambda) \subset \mathfrak{A}$
- such that $[\mathfrak{A}(\Lambda_1), \mathfrak{A}(\Lambda_2)] = 0$ when $\Lambda_1 \cap \Lambda_2 = \emptyset$
- $\bigcup_{\Lambda \subset S} \mathfrak{A}(\Lambda)$ is dense in $\mathfrak{A}$

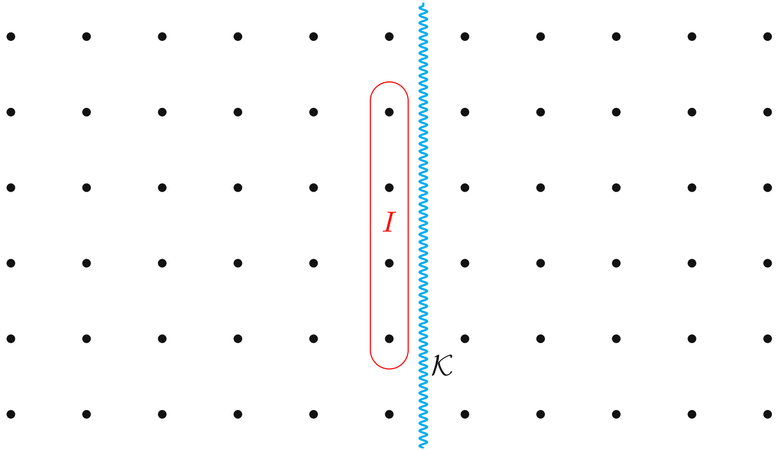
We do not assume $\mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$!

We further assume $\Lambda \mapsto p_\Lambda \in \mathfrak{A}(\Lambda)$ is a net of projections with $\Lambda \subset \Delta \Rightarrow p_\Delta \leq p_\Lambda$.

Example

Consider frustration-free local Hamiltonians $H_\Lambda = \sum_{\Delta\subset\Lambda} \Phi(\Delta)$. Then the projections p_Λ onto the (local) ground state space satisfy the conditions.

Quantum lattice systems: boundary cut



Question

What can we say about the bulk of the system by looking at the boundary?

Boundary algebras

Theorem

Consider $(\mathfrak{A}, p)$ and suppose the LTO axioms hold. Then we have the following:

- *There is a unique state ψ of $\mathfrak{A}$ such that $\psi(p_\Delta) = 1$. This state is pure.*
- *We can define a local net of boundary algebras $I \mapsto \mathfrak{B}(I)$.*

Note: in general $\mathfrak{B}(I)$ is not a subalgebra of $\mathfrak{A}(I)$.

Example

The conditions are satisified in the toric code, quantum double models, Levin–Wen models, and others. In each case we can compute the boundary algebra.

Bulk-boundary

In all known examples we have the following:

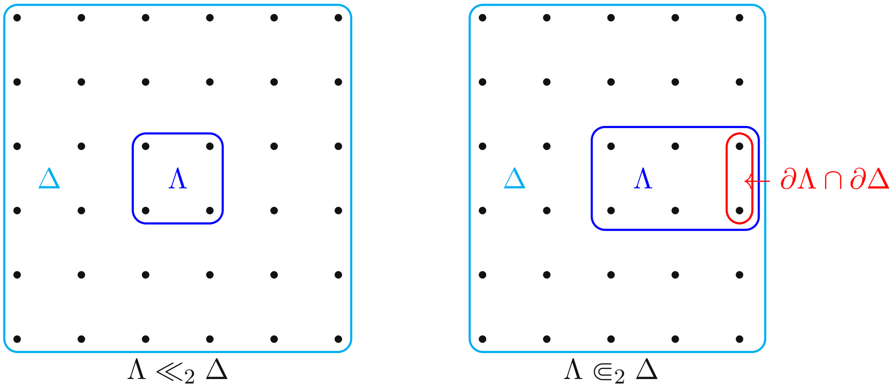
- The bulk (DHR) superselection theory is given by $Z(\mathcal{C})$ for some fusion category $\mathcal{C}$
- The boundary algebras are fusion categorical nets: $\mathfrak{B}(I) \cong \mathbf{End}_{\mathcal{D}}(X^{\otimes \# I})$ with $X = \bigoplus_{d \in \mathbf{Irr}(\mathcal{D})} d$.
- For such nets, one can compute the category of DHR bimodules[1], and it is equal to $Z(\mathcal{D})$
- $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, i.e. $Z(\mathcal{C}) \cong Z(\mathcal{D})$ as braided categories

The bulk topological order can be recovered from the boundary! This is a manifestation of topological holography.

[1] C. Jones, *Quantum Topology* **15** (2024)

LTO axioms

Surrounding rectangles



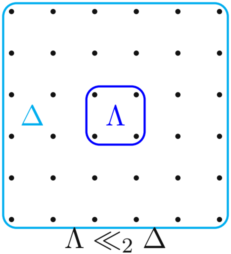
Note: $\partial\Lambda \cap \partial\Delta$ is the interval I from before.

First LTO axiom

LTO1

For all $\Lambda \ll_s \Delta$, we have $p_\Delta \mathfrak{A}(\Lambda) p_\Delta = \mathbb{C} p_\Delta$.

- This implies the Bravyi-Hastings-Michalakis LTQO axioms
- Reflects a quantum error correcting code property
- Get a state ψ by defining $\psi(a) = p_\Delta a p_\Delta$ for $a \in \mathfrak{A}(\Lambda)$.
- s is model-dependant, but should be uniform

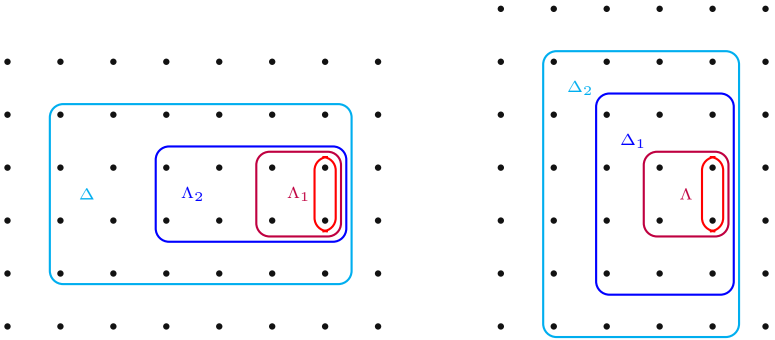


Remaining axioms

LTO2–LTO4 (paraphrased)

When $\Lambda \Subset_s \Delta$ with $I := \partial\Lambda \cap \partial\Delta$ there is a $\mathbf{C}^*$-algebra $\mathfrak{B}(I)$ such that $p_\Delta \mathfrak{A}(\Lambda) p_\Delta \cong \mathfrak{B}(I)$ (independent of Δ big enough). Moreover, operators in $\mathfrak{B}(I)$ commute with all $p_{\Delta'}$ such that $\Lambda \Subset_s \Delta'$ and with the same intersection I.

Note: $\mathfrak{B}(I)$ plays the role of $\mathbb{C}$ in LTO1.



Concluding remarks

- Can also consider models with topological boundaries
- Including higher-dimensional models (e.g. Walker–Wang)
- There is a canonical state on the boundary which has interesting properties
- Boundary algebra depends on choosing a side of the cut. Can give different algebras!
- They are however often related, which is important e.g. to proof Haag duality
- Can get a similar boundary algebra from reflection positivity and Osterwalder-Schrader reconstruction[2]

[2] Liu, Zhao, arXiv:2510.20662